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CONTENTS

SOME DEFINITE INTEGRALS OCCURRING IN HAVELOCK'S WORK ON THE WAVE RESISTANCE OF SHIPS

H. Bateman

1. In his study of the effect of varying draught[2] Havelock requires the values of integrals of the form

$$(1) \qquad \int_0^{\frac{\pi}{2}} e^{-\beta\sec^2\phi} \cos^{2m+1}\phi \, d\phi,$$

and the first two sections of this note are devoted to a study of such integrals. The substitution $\beta\tan^2\phi = t$ gives the equivalent expression

$$(2) \qquad \frac{1}{2\sqrt{\beta}}\, e^{-\beta}\int_0^\infty t^{-\frac{1}{2}}\left(1+\frac{t}{\beta}\right)^{-m-\frac{3}{2}} e^{-t}\, dt = \tfrac{1}{2}\sqrt{\pi}\, \beta^{\frac{1}{2}m}\, e^{-\frac{1}{2}\beta}\, W_{-\frac{1}{2}m-\frac{1}{2},\,-\frac{1}{2}m-\frac{1}{2}}(\beta),$$

where W is Whittaker's confluent hypergeometric function[3]. The asymptotic expansion of this is[4]

$$(3) \qquad \tfrac{1}{2}\sqrt{(\pi/\beta)}\, e^{-\beta}\, {}_2F_0\left(\tfrac{1}{2},\, m+\tfrac{1}{2};\, -\frac{1}{\beta}\right)$$

and this divergent series is also the asymptotic expansion of the integral

$$(4) \qquad \frac{\Gamma(\frac{1}{2})e^{-\beta}}{2\sqrt{\beta}\,\Gamma(m+\frac{3}{2})}\int_0^\infty t^{m+\frac{1}{2}}\left(1+\frac{t}{\beta}\right)^{-\frac{1}{2}} e^{-t}\, dt = \tfrac{1}{2}\sqrt{\pi}\,\beta^{\frac{1}{2}m} e^{-\frac{1}{2}\beta}\, W_{-\frac{1}{2}m-\frac{1}{2},\,\frac{1}{2}m+\frac{1}{2}}(\beta).$$

The two integrals (2) and (4) are equivalent on account of Kummer's relation[5]

$$(5) \qquad \frac{Z^b}{\Gamma(b)}\int_0^\infty s^{b-1}(1+s)^{-a} e^{-Zs}\, ds = \frac{Z^a}{\Gamma(a)}\int_0^\infty s^{a-1}(1+s)^{-b} e^{-Zs}\, ds$$

which was indicaded by his expression

$$(6) \qquad \frac{\Gamma(a-b)}{\Gamma(a)}\, Z^b\, {}_1F_1(b;\, b-a+1;\, Z) + \frac{\Gamma(b-a)}{\Gamma(b)}\, Z^a\, {}_1F_1(a;\, a-b+1;\, Z)$$

for either side of (5). This expression requires modification when a and b differ by an integer and so an alternative proof of (5) may be useful[6].

2. It will be assumed that a and b have values for which the integrals (5) have a meaning. Then with a suitable choice of c the equation may be written in the form

$$(7)\quad \Gamma(a)\Gamma(c-a)\int_0^\infty e^{-Zt}\, t^{c-b-1}\, dt \int_0^\infty e^{-Zs}\, s^{b-1}(1+s)^{-a}ds =$$

$$\Gamma(b)\Gamma(c-b)\int_0^\infty e^{-Zt} t^{c-a-1} dt \int_0^\infty e^{-Zs} s^{a-1}(1+s)^{-b}ds.$$

Now, Pareto[7] has pointed out that if

$$F(Z) = \int_0^\infty e^{-Zt} f(t)dt \quad \text{and} \quad G(Z) = \int_0^\infty e^{-Zt} g(t)dt \quad \text{then}$$

$$F(Z)\, G(Z) = \int_0^\infty e^{-Zt} h(t)dt$$

where $h(t)$ is expressed by the convolution integral[8]

$$h(t) = \int_0^t f(t-u)g(u)du,$$

so that the relation (7) holds if

$$\Gamma(a)\Gamma(c-a)\int_0^t (t-u)^{c-b-1}u^{b-1}(1+u)^{-a}du = \Gamma(b)\Gamma(c-b)\int_0^t (t-u)^{c-a-1}u^{a-1}(1+u)^{-b}du,$$

i.e. if ${}_2F_1(a,b;c;-t) = {}_2F_1(b,a;c;-t)$.

Kummer's relation (5) is thus implied by the well-known property of the interchangeability of the parameters of the first kind in Euler's hypergeometric function.

3. In his second approximation for the case of a circular cylinder in a uniform stream[10] Havelock used the two integrals

$$(8)\quad L_\tau = \int_0^{\frac{\pi}{2}} \cos(2\tau\phi - k\tan\phi)d\phi$$

$$(k > 0)$$

$$(9)\quad M_\tau = \int_0^{\frac{\pi}{2}} \sin(2\tau\phi - k\tan\phi)d\phi$$

for which values were given for $\tau = 0, 1, 2, 3, 4, 5, 6$. The rest of this note contains come observations on these integrals.

The integral L_τ has been studied elsewhere [11]. A general expression for positive integral values of τ is

$$(10)\quad L_\tau = (-)^{\tau-1}\pi e^{-k}\, {}_1F_1(1-\tau;2;2k)$$

while L_0 has Laplace's value $\frac{1}{2}\pi e^{-k}$.

We have

(11) $M_0 = -\int_0^{\frac{\pi}{2}} \sin(k\tan\theta)d\theta = -\int_0^{\infty} \frac{\sin\, kt}{1+t^2}\, dt = -\frac{1}{2}\{e^{-k}li(e^k) - e^k li(e^{-k})\}$.

where

$$li(e^z) = \int_0^{e^z} \frac{du}{\log u} = \gamma + \log z + \frac{z}{1.1!} + \frac{z^2}{2.2!} + \frac{z^3}{3.3!} + \cdots$$

is the logarithmic integral (γ is Euler's constant). When use is made of a recent result obtained by Copson [12], M_0 can be expressed in the form

(12) $M_0 = \quad shk \log k - \sum_{n=0}^{\infty} \frac{k^{2n+1}}{\Gamma(2n+2)} \psi\,(2n+2)$,

where $\psi\,(z) = \frac{d}{dz} \log \Gamma(z)$. Havelock's expression for M_τ is of the type

(13) $M_\tau = -\frac{1}{\pi} L_\tau li(e^k) + R_{\tau-1}(k)$

where $R_{\tau-1}$ is a polynomial of degree τ-1. There will also be an expansion

(14) $M_\tau = -.\frac{1}{\pi} L_\tau \log k + \sum_{n=0}^{\infty} A_n \frac{k^n}{n!}$

in which the coefficeents A_n have values which are readily found by means of a recurrence formula.

In the case of the function L_τ it is known that [13]

(15) $2kL_\tau = 2\tau L_\tau + (\tau - 1)\, L_{\tau-1} + (\tau + 1)\, L_{\tau+1}$.

The corresponding relation for M_τ seems to be

(16) $2kM_\tau + 2 = 2\tau M_\tau + (\tau - 1)\, M_{\tau-1} + (\tau + 1)\, M_{\tau+1}$.

This relation may be checked with the aid of Havelock's values

(17) $R_0 = 1,\ R_1 = k,\ R_2 = \frac{1}{3}(1 - 4k + 2k^2),\ R_3 = \frac{1}{3}k(5 - 5k + k^2)$.

The coefficients A_n may also be derived with the aid of Archibald's solution of the confluent hypergeometric equation in the logarithmic case [14] in the following manner. The differential equations satisfied by L_τ and M_τ are

(18) $k\frac{d^2L_\tau}{dk^2} + (2\tau - k)L_\tau = 0, \qquad k\frac{d^2M_\tau}{dk^2} + (2\tau - k)M_\tau = 1.$

The first of these equations may be reduced to the equation of the confluent hypergeometric function by the substitution $L_\tau = kc^{-k}\, Z_\tau$. The second equation has a particular integral of the type

$$\frac{1}{2\tau} + a_2 \frac{k^2}{2!} + a_3 \frac{k^3}{3!} + \cdots$$

when $\tau \neq 0$, and the general solution is obtained by adding the general solution of the equation for L_τ. When τ is a negative integer, $-s$ say, the recurrence relations (15) and (16) indicate that $L_\tau = 0$ and

$$(19) \qquad M_\tau = \frac{1}{\pi} li(e^{-k}) L_s(-k) - R_{s-1}(-k) \qquad (\tau = -s).$$

It should be noticed that both L_τ and M_τ satisfy the relations

$$(20) \qquad 2k \frac{dM_\tau}{dk} = (\tau - 1) M_{\tau-1} - (\tau + 1) M_{\tau+1}$$

$$(21) \qquad \frac{dM_\tau}{dk} + \frac{dM_{\tau+1}}{dk} = M_\tau - M_{\tau+1}$$

FOOTNOTES

1 Edited from a manuscript found among the papers of the late Professor Harry Bateman. The original manuscript has been followed as closely as possible, but some parts have been re-written, and a few references added. A. Erdélyi.

2 T. H. Havelock, Proc. Royal Soc. London A 108 (1925) 582-591

3 E. T. Whittaker and G.N. Watson, Modern Analysis § 16.12

4 Modern Analysis § 16.3

5 E. E. Kummer, J. für Math. 17 (1837) 228-242

6 In Havelock's paper m is an integer.

7 V. Pareto, J. für Math. 110 (1892) 290-323

8 This type of integral may, perhaps, be named for Johann Bernoulli on account of its occurrence in the problems of the tautochrone and brachistochrone. A special integral of this type occurred, however, in the work of John Wallis on the quadrature of the circle. Related special integrals, known as binomial integrals, occurred also in the work of Newton, Leibniz, and Euler. Integrals of the type under consideration occur also in the expression for the remainder in Taylor's theorem and in the Liouville-Riemann expression for a fractional integral. The use of integrals of Bernoulli's type in mathematical physics dates chiefly from the time of Poisson who used such integrals in the theory of hysteresis and other phenomena. Poisson was, I think, also the first to use the method of the inverse Laplace transformation in the solution of a physical problem (Cf. J.de l'Ecole Polytechnique t. 12 cah. 19 (1815) 1-162).

9 *Note by A.E.* E.G.C. Poole (Quart. J. of Math. Oxford 8)1938) 230-233) has shown that the property in question of the hypergeometric function can be established by integrations by parts when a and b differ by an integer, and the same is true of Kummer's relation (5). If $b - a$ is a positive integer, n say, the left hand side of (5) is

$$\frac{(-)^n Z^a}{\Gamma(a+n)} \int_0^\infty \frac{d^n e^{-Zs}}{ds^n} s^{a+n-1} (1+s)^{-a} \, ds = \frac{Z^a}{\Gamma(a+n)} \int_0^\infty e^{-Zs} \frac{d^n}{ds^n} \{s^{a+n-1} (1+s)^{-a}\} \, ds,$$

by n successive integrations by parts, and this is identical with the right hand side of (5).

10 T. H. Havelock, Proc. Royal Soc. London A 115 (1926) 268-280

11 H. Bateman, Trans. Amer. Math. Soc. 33 (1931) 817-831; Proc. Nat. Ac. Sciences Washington 12 (1931) 689-690

12 E. T. Copson, Proc. Camb. Phil. Soc. 37 (1941) 102-104

13 H. Bateman, l.c See also M. Lerch, J. für Math. 130 (1905) 47-65; N.G. Shabde, Bull. Calcutta Math. Soc. 24 (1932) 109-134; N.A. Shastri, Phil. Mag. (7) 20 (1935) 468-478; J. Indian Math. Soc. (N.S.) 3 (1938) 8-18, 152-154, 155-163.

14 W. J. Archibald, Phil. Mag. (7) 26 (1938) 415-419.

UNDER CERTAIN GROUPS

Sr. M. Philip Steele and V. O. McBrien

FOREWORD

Many teachers of mathematics like to introduce the concept of a Group at an early stage of the student's training so that, in more advanced study, there will be less confusion about this important idea. One common method which appeals to many freshmen is a consideration of the roots of unity. Thus, the equation $x^3 = 1$ has three roots, namely $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$; or, as they are usually called, 1, ω, and ω^2. This set of three numbers have the interesting property that the product of any two numbers of the set always gives a member of the set. For example, $\omega \cdot \omega^2 = 1$ and $\omega^2 \cdot \omega^2 = \omega$. This set of three is said to be *closed* under the operation of multiplication. This property of closure is a fundamental property of a mathematical group. The operation under which a group is closed need not be ordinary multiplication. For instance, in the following paper the operation of substitution is used frequently.

We may obtain a geometrical configuration of the group of three numbers 1, ω, and ω^2 by plotting them in the ordinary complex plane. The three numbers all lie on a circle of radius equal to unity and the vectors to these numbers are $120°$ apart. Then if ω^2 is multiplied by ω we see that this is equivalent to rotating the vector to ω^2 through $120°$ since $\omega^2 \cdot \omega = 1$. However, when the number 1 of the set is multiplied by any member of the set, that member remains unchanged. The number 1 is said to be the identity element of the group. In every group there is an identity element.

In the following paper we shall consider some configurations of certain groups in a manner somewhat analagous to the way described above for the three cube roots of unity. The methods used are the methods of inversive geometry; an inversive property being a geometrical property which is unchanged, or invariant, with respect to a set of transformations called direct circular transformations. An example of such a transformation is the reflection in the X-axis, $z' = \bar{z}$, where $\bar{z} = x - yi$ is the conjugate of z. We use the reflection in high school geometry when we turn a triangle over on an edge. The groups under discussion are the so-called finite inversive groups which are formed by the products of certain inversive transformations. The method demonstrates how the plane may be divided into a set of regions by a system of circles. No attempt is made in this paper to give a detail-

ed discussion of the nature of the inversive plane but it is to be noted that, throughout the paper, a straight line is considered to be a special case of a circle with an infinite radius.

1. *Introduction.* The purpose of this paper is to show how certain curves, invariant under the generating transformations of a dihedral group of order $2n$, divide the real inversive plane into $2n$ fundamental regions. The method and the results following seem to contribute to the classical method of building up finite inversive groups by forming the products of a set of transformations. [1, p. 75.]

We define here the term fundamental region in the same way as in the theory of the complex variable. Taking any point of the complex plane of Riemann, or the real inversive plane, and applying to it all the substitutions of the group, we obtain a set of equivalent points. Each fundamental region of the group contains one point, and not more than one point, which is equivalent with respect to the group of substitutions. [2, p. 290.] In what follows we shall have a finite number of regions; the system of curves by which the plane is divided into fundamental regions is defined as *the basic configuration of the plane under the group.*

2. *The Generating Transformations.* In considering the generating transformations below we make use of the one-to-one correspondence between the real inversive geometry of the plane and the complex projective geometry of the line. That is, to the complex points $z = x + yi$ of the line correspond the real points $z = x + yi$ of the plane, and to the complex projective transformations of the line correspond the real, direct, circular transformations of the plane. [3, p. 395].

For generators of the group we employ $R(\lambda) = \frac{r}{\lambda}$ and $S(\lambda) = s - \lambda$ where r, s, and λ are, in general, of the form $a + bi$. Both of these generators are of period two, and it has been shown [4, p. 424] that the group generated by them is the dihedral rotation group whose order is twice the order of the product (SR). It has also been proved [5, p. 80] that only four distinct groups can be generated by R, S, when r and s are rational numbers, but other groups of even order may be generated when r and s are complex numbers.

In the real inversive plane, the transformations are R, $z = \frac{r}{z'}$ and S, $z = s - z'$. They are not projective but are direct circular transformations, or Mobius transformations [3, p. 384]. Both R and S are the product of two inversions. The conditions on r and s, so that the group generated by R and S be of finite order has been discussed [1, pp. 80-87 and 6, pp. 3-7], and it has been shown that to generate a finite group, the ratio $\frac{s^2}{r}$ must be real and satisfy a certain polynomial equation.

3. *Curves of Basic Configuration for* G_{2n}. If, for the pair of

generators $R(\lambda)$ and $S(\lambda)$, $s = 0$, then (SR) is of order two, *i.e.* $(SR)^2$ = identity, and we obtain the only Abelian rotation group G_4. The transformations in the real inversive plane are then R, $z = \frac{r}{z'}$ and S, $z = -z'$.

The elements of the group are λ, $\frac{r}{\lambda}$, $-\frac{r}{\lambda}$, $-\lambda$, or in terms of the generators, I, R, $RS = SR$, S. We may regard every point in the plane as affected by each operation of the group. That is, each operation affects a transformation of all the points of the plane. However, certain values of λ yield less than four distinct points under the operations of the group. Thus, by setting λ equal to each of the other elements we obtain the three pairs of *invariant dyads*, $\pm r^{\frac{1}{2}}$, $(0,\infty)$, and $\pm ir^{\frac{1}{2}}$. By an invariant dyad is meant that each element is left unchanged or transformed into the other one under the operations of the group.

From the invariant sets of points we may easily build up the basic configuration of the plane under G_4. Since we are in the real inversive plane we shall regard all straight lines as proper circles. The circular transformations R and S send circles into circles. Further, R, $z' = \frac{r}{z}$, sends the point O into ∞ and vice versa and S, $z' = -z$, sends circles into circles by reflecting them in the origin. The invariant dyad $(0,\infty)$ lies on the join of $\pm r^{\frac{1}{2}}$, namely:

$$r^{-\frac{1}{2}}z - r^{\frac{1}{2}}\bar{z} = 0$$

Likewise, $(0,\infty)$ lies on the join of $\pm ir^{\frac{1}{2}}$, namely:

$$r^{-\frac{1}{2}}z + r^{\frac{1}{2}}\bar{z} = 0$$

Also, the four points, $\pm r^{\frac{1}{2}}$, $\pm ir^{\frac{1}{2}}$, lie on the circle with center at O and radius $= |\sqrt{r\bar{r}}|$, namely:

$$z\bar{z} = (r\bar{r})^{\frac{1}{2}}$$

All applications of the G_4 leave these three circles invariant. Hence we call the three curves

$$r^{-\frac{1}{2}}z - r^{\frac{1}{2}}\bar{z} = 0$$

$$r^{-\frac{1}{2}}z + r^{\frac{1}{2}}\bar{z} = 0$$

$$z\bar{z} = (r\bar{r})^{\frac{1}{2}}$$

the basic configuration of the plane under G_4.

We have already mentioned that r may be real. This is clear, since if $r = 1$, we have

$$z = \bar{z}$$

$$z = -\bar{z}$$

$$z\bar{z} = 1$$

The transformation $z' = rz$ merely stretches the plane from the origin in the ratio $|r|:1$ and rotates the plane through an angle equal to the amplitude of r. Thus, our basic configuration is changed only in size and orientation with respect to the axis of reals. Figure 1 shows the basic configuration for real r.

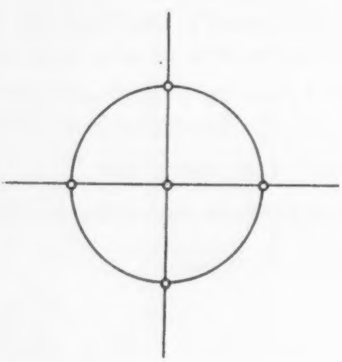

Figure 1

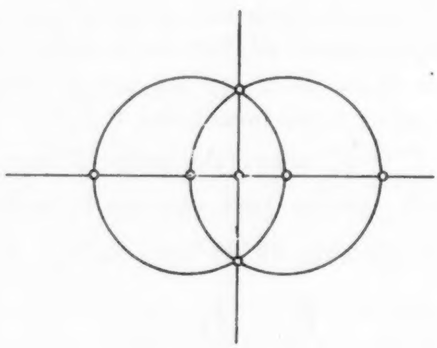

Figure 2

Under the G_4 there are four fundamental regions cut out by the basic configuration because if we apply the transformations R, RS, and S to a point λ in the plane we obtain a set of equivalent points. In this sense any two adjacent sections of the eight sections cut out by the configuration form a fundamental region. That there are eight sections suggests the well known property that this group is a subgroup of the inversive group of the rectangle or of the rhombic four point [1, pp. 82-83].

When $s^2 = r$, the group generated is of order six since $(SR)^3 = 1$. The basic configuration of this group has been discussed [7, p. 24]. This group is isomorphic with the well-known cross ratio group. The generators are $R(\lambda) = \frac{s^2}{\lambda}$ and $S(\lambda) = s - \lambda$, where $s \neq 0$. The elements are, respectively, I, RS, SR, S, $RSR = SRS$, R, or λ, $\frac{s^2}{(s - \lambda)}$, $\frac{(s\lambda - s^2)}{\lambda}$,

$s - \lambda$, $\frac{(s\lambda)}{(\lambda - s)}$, $\frac{s^2}{\lambda}$. Setting λ equal to each of the elements of the of the G_6 we obtain the invariant sets of conjugate points, $(-s, 2s, \frac{s}{2})$., $(s, 0, \infty)$, $(-s\omega^2, -s\omega)$ where $\omega^2 + \omega + 1 = 0$.

The only invariant line under the group is the join of O and $\frac{s}{2}$,

$$\bar{s}z - s\bar{z} = 0$$

Both invariant triads lie on this line. The other curves of the configuration are: the circle with center at O and radius $|s|$, the circle with center at s and radius $|s|$, and the join of $(-s\omega^2, -s\omega)$. These circles are:

$$z\bar{z} = s\bar{s}$$

$$(z - s)(\bar{z} - \bar{s}) = s\bar{s}$$

$$\bar{s}z - s\bar{z} - s\bar{s} = 0$$

If $s = 1$, we have the invariant system of $(n + 1)$ circles (Fig. 2)

$$z\bar{z} = 1$$

$$z + \bar{z} = 1$$

$$(z - 1)(\bar{z} - 1) = 1$$

$$z - \bar{z} = 0$$

The twelve sections cut out by the configuration suggests the property that it consists of those operations of the inversive group, G_{12}, which are formed by an even number of inversions.

The manner in which the basic configuration is built up is similar for the groups of order eight and twelve.

To generate the octic group, G_8, we have, when $s^2 = 2r$, $(RS)^4 = 1$. Hence, the generators are $R(\lambda) = \frac{s^2}{2\lambda}$ and $S(\lambda) = s - \lambda$, or, in the inversive plane.

$$R: \quad z' = \frac{s^2}{2z} \qquad S: \quad z' = s - z$$

The elements of the G_8 are λ, $s - \lambda$, $\frac{s^2}{2\lambda}$, $\frac{s^2}{(2s - 2\lambda)}$, $\frac{(2s\lambda - s^2)}{2\lambda}$, $\frac{(s^2 - 2s\lambda)}{(2s - 2\lambda)}$, $\frac{(s^2 - s\lambda)}{(s - 2\lambda)}$, $\frac{(s\lambda)}{(2\lambda - s)}$ or I, S, R, SR, RS, SRS, $RSRS$, RSR. There are just ten points in the plane which are invariant under the G_8 (*i.e.*, they are transformed into less than 8 distinct points by G_8). These are composed of the two sets of four conjugates $(s, \frac{s}{2}, 0, \infty)$,

$\left[\pm\frac{s\sqrt{2}}{2}, s(1 \pm \frac{\sqrt{2}}{2})\right]$, and the pair of conjugates $\frac{s(1 \pm i)}{2}$. The two sets of four conjugate points lie on the join of O and $\frac{s}{2}$, the only invariant line in the real inversive plane under G_8. The equation of the line (circle) is

$$\bar{s}z - s\bar{z} = 0$$

The join of the conjugate pair $\frac{s(1 \pm i)}{2}$ is

$$\bar{s}z + s\bar{z} - s\bar{s} = 0$$

This line passes through $\frac{s}{2}$, one of the other invariant points.

As in the G_6, the invariant points which lie on the only invariant line are related by pairs to the points *not* on the invariant line, $\frac{s(1 \pm i)}{2}$. Thus, the circle on O and s and radius $\frac{1}{2}|s|$ passes through $\frac{s(1 \pm i)}{2}$. The equation of this circle is

$$(z - \frac{s}{2})(\bar{z} - \frac{\bar{s}}{2}) = \frac{(s\bar{s})}{4}$$

Likewise the pair of circles on $\pm s\frac{\sqrt{2}}{2}$ and $s(1 \pm \frac{\sqrt{2}}{2})$ with radii equal to $\frac{\sqrt{2}}{2}|s|$ also pass through the above pair. The equations of these circles are respectively:

$$z\bar{z} = \frac{s\bar{s}}{2}$$

and

$$(z - s)(\bar{z} - \bar{s}) = \frac{s\bar{s}}{2}$$

The invariant line (the join of O and $\frac{s}{2}$), along with n (in this case, four) circles of the coaxial system of circles with axis as the join of $\frac{s(1 \pm i)}{2}$ are invariant under all the operations of G_8. We thus have a basic configuration of the plane under G_8 consisting of the system of circles:

$$\bar{s}z - s\bar{z} = 0$$

$$\bar{s}z + s\bar{z} - s\bar{s} = 0$$

$$(z - \frac{s}{2})(\bar{z} - \frac{\bar{s}}{2}) = \frac{s\bar{s}}{4}$$

$$z\bar{z} = \frac{s\bar{s}}{2}$$

$$(z - s)(\bar{z} - \bar{s}) = \frac{s\bar{s}}{2}$$

The basic configuration is shown in Fig. 3 where the value of s is taken to be on the axis of reals (in this case we let s equal two for convenience). There are eight fundamental regions, the basic configuration cutting out sixteen sections in the plane.

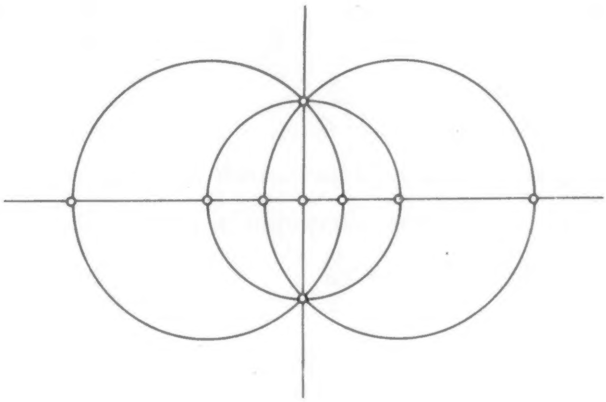

Figure 3

If $s^2 = 3r$ then $(RS)^6 = 1$, we obtain the dihedral rotation group, G_{12}, with elements λ, $s - \lambda$, $\frac{s^2}{3\lambda}$, $\frac{s^2}{(3s - 3\lambda)}$, $\frac{(3s\lambda - s^2)}{3\lambda}$, $\frac{s\lambda}{(3\lambda - s)}$, $\frac{(2s\lambda - s^2)}{(3\lambda - s)}$, $\frac{(3s\lambda - s^2)}{(6\lambda - 3s)}$, $\frac{(3s\lambda - 2s^2)}{(6\lambda - 3s)}$, $\frac{(2s\lambda - s^2)}{(3\lambda - 2s)}$, $\frac{(s\lambda - s^2)}{(3\lambda - 2s)}$, $\frac{(2s^2 - 3s\lambda)}{(3s - 3\lambda)}$, or I, S, R, SR, RS, RSR, $RSRS$, $RSRSR$, $SRSRSR = RSRSRS$, $SRSRS$, $SRSR$, SRS. The transformations are $z' = \frac{s^2}{3z}$ and $z' = s - z$, both being the products of a pair of inversions.

The invariant elements consist of the invariant dyad $\left[\frac{s}{2} \pm is\frac{\sqrt{6}}{6}\right]$ and the two sets of six conjugates $(0, \infty, s, \frac{s}{3}, \frac{2s}{3}, \frac{s}{2})$, $(\pm s\frac{\sqrt{3}}{3}, s \pm s\frac{\sqrt{3}}{3}, \frac{s}{2} \pm s\frac{\sqrt{3}}{6})$.

The only invariant line under G_{12} is the join of O and s

which line contains all the points of two sets of six conjugates which are invariant under the operations of the G_{12}. Again we have a system of co-axial circles with axis as the join of the invariant pair $\left(\frac{s}{2} \pm is\frac{\sqrt{6}}{6}\right)$ which has the equation

$$\bar{s}z - s\bar{z} - s\bar{s} = 0$$

As in the group of orders six and eight it is easily verified that this line passes through $\frac{s}{2}$.

The circle on $\left(\frac{s}{2} \pm s\frac{\sqrt{3}}{6}\right)$ which passes through the invariant dyad $\left(\frac{s}{2} \pm is\frac{\sqrt{3}}{6}\right)$ has for its center $\frac{s}{2}$ and radius $\frac{\sqrt{3}}{6}|s|$. Its equation is

$$(z - \frac{s}{2})(\bar{z} - \frac{\bar{s}}{2}) = \frac{s\bar{s}}{12}$$

We have also the pair of invariant circles on $(0, \frac{2s}{3})$ and $(\frac{s}{3}, s)$, respectively, which pass through the invariant dyad

$$(z - \frac{s}{3})(\bar{z} - \frac{\bar{s}}{3}) = \frac{s\bar{s}}{9}$$

$$(z - \frac{2s}{3})(\bar{z} - \frac{2\bar{s}}{3}) = \frac{4s\bar{s}}{9}$$

The pair of circles on $\pm s\frac{\sqrt{3}}{3}$ and $(s \pm s\frac{\sqrt{3}}{3})$ with centers at O and s also belong to the coaxial invariant system. They are, respectively,

$$\left(z - s\frac{\sqrt{3}}{3}\right)\left(\bar{z} - \bar{s}\frac{\sqrt{3}}{3}\right) = \frac{s\bar{s}}{3}$$

$$\left(z - \left(s + s\frac{\sqrt{3}}{3}\right)\right)\left(\bar{z} - \left(\bar{s} + \bar{s}\frac{\sqrt{3}}{3}\right)\right) = 2s\bar{s}\left(1 + \frac{\sqrt{3}}{3}\right)$$

Thus, the real inversive plane is divided into 12 fundamental regions by the basic configuration under G_{12} consisting of the invariant set of $(n + 1)$ circles (Fig. 4)

$$\bar{s}z - s\bar{z} = 0$$

$$\bar{s}z + s\bar{z} - s\bar{s} = 0$$

$$(z - \frac{s}{2})(\bar{z} - \frac{\bar{s}}{2}) = \frac{s\bar{s}}{12}$$

$$(z - \frac{s}{3})(\bar{z} - \frac{\bar{s}}{3}) = \frac{s\bar{s}}{9}$$

$$(z - \frac{2s}{3})(\bar{z} - \frac{2\bar{s}}{3}) = \frac{4s\bar{s}}{9}$$

$$\left[z - s\frac{\sqrt{3}}{3}\right]\left[\bar{z} - \bar{s}\frac{\sqrt{3}}{3}\right] = \frac{s\bar{s}}{3}$$

$$\left[z - \left[s + s\frac{\sqrt{3}}{3}\right]\right]\left[\bar{z} - \left[\bar{s} + \bar{s}\frac{\sqrt{3}}{3}\right]\right] = 2s\bar{s}\left[1 + \frac{\sqrt{3}}{3}\right]$$

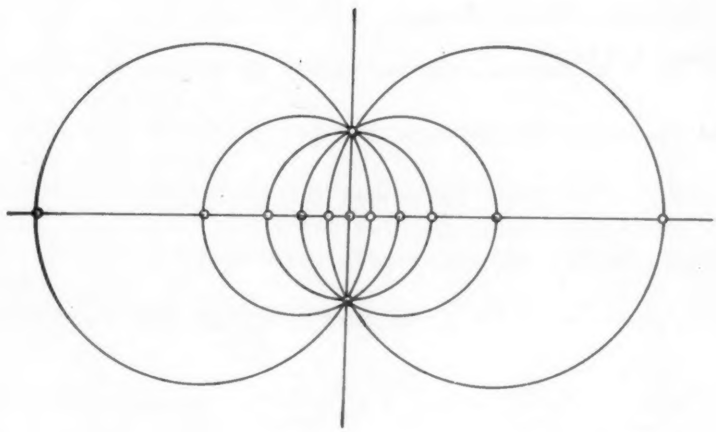

Figure 4

This method of finding the basic configuration of the real inversive plane is perfectly general for all dihedral rotation groups, for all such groups may be represented as a group of subtraction and division when $\frac{s^2}{r}$ may be irrational [5, p. 85 and 6, p. 4]. For example, the dihedral G_{10} has a basic configuration in the real inversive plane, but in this case $\frac{s^2}{r}$ is irrational. Furthermore, the geometry of the configuration leads to the suggestion that some of the invariant points are notable points of regular n-gons.

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COLLEGIATE ARTICLES

Graduate Training not required for Reading

ON THE SQUARES OF SOME TRIANGULAR NUMBERS

Pedro A. Piza

Since ancient times the following theorem has been known:

The square of the triangular number of order x is equal to the sum of the first x cubes.

The expression for this classic theorem is

$$\left[\frac{x(x+1)}{2}\right]^2 = \sum_{a=1}^{x} a^3,$$

which can easily be proved by induction upon adding $(x+1)^3$ to each side of the equation.

I have found the following related theorem, the proof of which is the object of this note:

> The square of the triangular number of order $(x^2 + x)$ is equal to twice the sum of the first x cubes, plus **five** times the sum of the first x fifth powers, plus twice the **sum of** the first x seventh powers.

The equation expressing this theorem is

$$\left[\frac{(x^2+x)(x^2+x+1)}{2}\right]^2 = 2\sum_{a=1}^{x} a^3 + 5\sum_{a=1}^{x} a^5 + 2\sum_{a=1}^{x} a^7 = \sum_{a=1}^{x^2+x} a^3. \qquad (1)$$

Before attempting a proof of (1) we shall establish the following general relation:

$$\frac{(x^2+x)^n}{2} = \sum_{c=1}^{\infty} \binom{n}{2c-1} \sum_{a=1}^{x} a^{2n+1-2c}. \qquad (2)$$

When $x = 1$, $\sum_{a=1}^{\infty} a^m = 1$ for all values of m. Hence

$$\frac{2^n}{2} = \sum_{c=1}^{\infty} \binom{n}{2c-1} = 2^{n-1},$$

which is a known property of binomial coefficients.

Suppose that (2) is true when $x = y - 1$, $y > 1$, so that

$$\frac{(y-1)^n y^n}{2} = \frac{(y^2-y)^n}{2} = \sum_{c=1}^{\infty} \binom{n}{2c-1} \sum_{a=1}^{y-1} a^{2n+1-2c}.$$

Add the identity

$$\frac{(y^2+y)^n-(y^2-y)^n}{2}=\sum_{c=1}^{\infty}\binom{n}{2c-1}y^{2n+1-2c}$$

to obtain

$$\frac{(y^2+y)^n}{2}=\sum_{c=1}^{\infty}\binom{n}{2c-1}\sum_{a=1}a^{2n+1-2c}$$

which completes the proof of (2) for all values of x and n, by mathematical induction.

For $n = 2$, 3, and 4, we have by (2)

$$\frac{(x^2+x)^2}{2}=2\sum_{a=1}^{x}a^3$$

$$\frac{(x^2+x)^3}{2}=3\sum_{a=1}^{x}a^5+\sum_{a=1}^{x}a^3$$

$$\frac{(x^2+x)^4}{2}=4\sum_{a=1}^{x}a^7+4\sum_{a=1}^{x}a^5$$

The first member of (1) can be written

$$\frac{[(x^2+x)^2+(x^2+x)]^2}{4}=\frac{(x^2+x)^4}{4}+\frac{(x^2+x)^3}{2}+\frac{(x^2+x)^2}{4}$$

Therefore we write

$$\frac{(x^2+x)^4}{4}=2\sum_{a=1}^{x}a^7+2\sum_{a=1}^{x}a^5$$

$$\frac{(x^2+x)^3}{2}=3\sum_{a=1}^{x}a^5+\sum_{a=1}^{x}a^3$$

$$\frac{(x^2+x)^2}{4}=\sum_{a=1}^{x}a^3$$

And adding the above three equations we obtain:

$$\left[\frac{(x^2+x)(x^2+x+1)}{2}\right]^2=2\sum_{a=1}^{x}a^3+5\sum_{a=1}^{x}a^5+2\sum_{a=1}^{x}a^7.$$

Q. E. D.

SOME EXAMPLES ILLUSTRATING CONTINUITY AND DIFFERENTIABILITY

Gordon Raisbeck*

FOREWORD

The concepts of continuity and differentiability of a real function of a single real variable are not generally understood even by people whose daily work brings them into contact with calculus.

Let us first review the definitions of continuity and differentiability. A function $f(x)$ of a single real variable x is said to be continuous at a point x_0 if for every number $\epsilon > 0$ there exists a number $\delta > 0$ such that

(1) $$|f(x) - f(x_0)| < \epsilon \quad \text{for every } x \text{ such that } |x - x_0| < \delta$$

An equivalent definition is this: $f(x)$ is continuous at x_0 if

(2) $$\lim_{x \to x_0} f(x) = f(x_0)$$

A function $f(x)$ is said to have a derivative at the point x_0 if

(3) $$\lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta}$$

exists, and in that case the derivative is equal to the above limit.

A function is said to be continuous or to have a derivative in an interval if it is continuous or has a derivative at every point of the interval.

It would defeat the purpose of this article to give a rough idea or a description in words of what these definitions mean, since the principal examples to be shown are designed to illustrate cases where the rough ideas fail.

Introduction. Discontinuities are usually classified in three classes: removable discontinuities, simple discontinuities, and essential discontinuities. A function $f(x)$ is said to have a removable discontinuity at x_0 if

(4) $$\lim_{x \to x_0} f(x) = a \neq f(x_0)$$

As an example of such a function we may take

(5) $$f(x) = \lim_{n \to \infty} \sqrt[n]{|x|} = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

which has a removable discontinuity at $x = 0$. Such a discontinuity is called removable because it can be removed by redefining $f(x)$ at x_0 so that $f(x_0) = \lim_{x \to x_0} f(x)$.

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A function $f(x)$ has a simple discontinuity at x_0 if the following conditions exist:

(6) $$\lim_{x \to x_0 - 0} f(x) = a$$

and

(7) $$\lim_{x \to x_0 + 0} f(x) = b$$

but $a \neq b$. We say $x \to x_0 + 0$ if $x \to x_0$ in such a manner that $x \geq x_0$; and $x \to x_0 - 0$ if $x \to x_0$ in such a way that $x \leq x_0$. Equation 6 defines the left-hand limit of $f(x)$ at x_0, and equation 7 defines the right-hand limit of $f(x)$ at x_0. An example of a function with a simple discontinuity is the postage required on first-class mail considered as a function of weight. Here

$$f(x) = .03 \qquad 0 < x < 1$$

$$f(x) = .06 \qquad 1 \leq x < 2$$

and so forth. Here

(8) $$\lim_{x \to 1 - 0} f(x) = .03$$

but

(9) $$\lim_{x \to 1 + 0} f(x) = .06$$

and the function is discontinuous at $x = 1$.

A function is said to have an essential discontinuity at x_0 if either the right-hand or the left-hand limit as x approaches x_0 does not exist. An example of such a function is

(10) $$\begin{aligned} f(x) &= \sin \frac{1}{x} \qquad & x \neq 0 \\ &= 0 \qquad & x = 0 \end{aligned}$$

Here neither the right-hand nor the left-hand limit exists.

In this paper will be shown several examples of functions whose behavior is startling and seems paradoxical in the light of popular conceptions of continuity and differentiability.

Functions of Unusual Behavior. Let

(11) $$\begin{aligned} f(x) &= 1 \qquad & x \text{ rational} \\ &= 0 \qquad & x \text{ irrational.} \end{aligned}$$

For the benefit of those who believe that this function is excessively artificial, it may be pointed out that

(12) $f(x) = \lim_{m\to\infty} \lim_{n\to\infty} [\cos(m!\pi x)]^{2n}$

It is easy to see that

(13) $\lim_{x\to a} f(x)$

does not exist for any a; hence $f(x)$ has an essential discontinuity everywhere. Furthermore, this function has no derivative, because a function cannot have a derivative at a point where it is discontinuous.

The next illustration is a function which is discontinuous for every rational value of the argument, but which has a derivative for some (necessarily irrational) values. We assume the following lemma:

Lemma[1] If x is a rational number $\frac{p}{q}$ and a is a quadratic surd there is a positive number k depending on a but not on p and q such that

$$|a - x| > \frac{k}{q^2}$$

for all x.

Let $f(x)$ be the function defined thus:

(14) $f(x) = \frac{1}{q^3}$ if $x = \frac{p}{q}$ reduced to lowest terms and q is positive

$= 0$ if x is irrational.

If we recall that irrational numbers exist as close as we please to any rational number, it follows that $f(x)$ is not continuous for any rational value of x. On the other hand, if we let x_0 be a quadratic surd, then

(15) $\lim_{\delta\to 0} \frac{f(x_0 + \delta) - f(x_0)}{\delta} = \left[\frac{df}{dx}\right]_{x_0}$

exists. For if

(16) $x_0 + \delta = \frac{p}{q}$

(17) $\delta = \frac{p}{q} - x_0$

(18) $|\delta| \geq \frac{k}{q^2}$

by lemma, and

(19) $q \geq \frac{\sqrt{k}}{\sqrt{\delta}}$

[1]Hardy and Wright, *Theory of Numbers*, Thm. 188, p. 157

Therefore

$$(20) \qquad \left|\frac{f(x_0+\delta)-f(x_0)}{\delta}\right| \le \frac{\frac{1}{q^3}}{\frac{k}{q^2}} = \frac{1}{qk}$$

since $f(x_0) = 0$. Hence

$$(21) \qquad \lim_{\delta\to 0}\left|\frac{f(x_0+\delta)-f(x)}{\delta}\right| \le \lim_{\delta\to 0}\frac{1}{qk} \le \lim_{\delta\to 0}\frac{\sqrt{\delta}}{k\sqrt{k}} = 0$$

by 19 and 20. If on the other hand $x_0 + \delta$ is irrational then the limit 21 is also equal to 0. Therefore

$$(22) \qquad \left(\frac{df}{dx}\right)_{x_0} = 0 \quad \text{if } x_0 \text{ is any quadratic surd.}$$

Actually, the points at which $f(x)$ has a derivative are not exceptional, but on the contrary the points at which f is not differentiable form a very thin set compared to the set of points where it is differentiable. A proof of this statement, in fact, an exact description of what it means, is beyond the scope of the present article.

The next example is an example due to van der Waerden[1] of a function which is continuous but has no derivative anywhere. We shall build up this function as an infinite series of functions defined as follows:

Let $f_0(x)$ be the distance from x to the nearest integer. Let $f_1(x)$ be the distance from x to the nearsst fraction of the form $\frac{p}{10}$. In general let $f_n(x)$ be the distance from x to the nearest fraction of the form $\frac{p}{10^n}$. Analytically,

$$(23) \qquad f_n(x) = \min\left|x - \frac{p}{10^n}\right|$$

where p takes all integer values. Obviously $f_n(x)$ has a period $\frac{1}{10^n}$, rises to a maximum of $\frac{1}{2\cdot 10^n}$, and has a derivative of ± 1 except for values of x of the form $\frac{p}{2\cdot 10^n}$, where it has cusps. Observe that if (x_1, x_2) is an interval in which $f_n(x)$ has no cusps,

$$(24) \qquad f_n(x_2) - f_n(x_1) = \pm(x_2 - x_1).$$

[1] B. L. van der Waerden, *Ein einfaches Beispiel einer nichtdifferenzierbar stetigen Function*, Math. Zeitschrift, 32, 1930

Now consider

$$(25)\qquad \sum_{n=1}^{\infty} f_n(x)$$

For any x

$$(26)\qquad |f_n(x)| \leq \frac{1}{2\cdot 10^n}$$

Hence the series 25 converges uniformly and we may say

$$(27)\qquad f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Since the series 25 converges uniformly, $f(x)$ is continuous.

Let us suppose for a moment that $f(x)$ has a derivative, *i.e.*, that

$$(28)\qquad \lim_{\delta\to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

exists. Then if we take any sequence of numbers $\delta_1, \delta_2, \ldots, \delta_n, \ldots$ such that

$$(29)\qquad \lim_{n\to\infty} \delta_n = 0$$

it will also be true that

$$(30)\qquad \lim_{n\to\infty} \frac{f(x+\delta_n) - f(x)}{\delta_n}$$

will exist also, and be equal to 28. We shall now find such a sequence $\delta_1, \delta_2, \ldots, \delta_n, \ldots$ for which the limit 30 does not exist.

Let us suppose that $0 \leq x < 1$. Divide the interval $(0,1)$ into $2\cdot 10^n$ equal intervals, with endpoints at the points $\frac{p}{10^n}$. We may say

$$(31)\qquad \frac{p}{2\cdot 10^n} \leq x < \frac{p+1}{2\cdot 10^n}$$

where p is an integer. Notice that the interval $\left[\frac{p}{2\cdot 10^n}, \frac{p+1}{2\cdot 10^n}\right]$ is a region inside which $f_n(x)$, $f_{n-1}(x)$, $\ldots$, $f_1(x)$, and $f_0(x)$ have no cusps.

Now consider the two points $x + 10^{-n-1}$ and $x - 10^{-n-1}$. One of these points is in the above interval. If the former is in the interval, let $\delta_n = 10^{-n-1}$. Otherwise, let $\delta_n = -10^{-n-1}$. Note that $\lim_{n\to\infty} \delta_n = 0$.

Let us return to

$$(32) \qquad \frac{f(x+\delta)-f(x)}{\delta} = \frac{\sum_{m=0}^{\infty}[f_m(x+\delta)-f_m(x)]}{\delta}$$

Let $\delta = \delta_n$ and divide the sum into two parts thus:

$$\frac{\sum_{m=0}^{n}[f_m(x+\delta_n)-f_m(x)]}{\delta_n} + \frac{\sum_{m=n+1}^{\infty}[f_m(x+\delta_n)-f_m(x)]}{\delta_n}$$

By equation 24, every term of the first sum is $\pm\delta_n$, and because of the periodicity of $f_m(x)$ every term of the second sum is zero. Hence

$$(33) \qquad \frac{f(x+\delta_n)-f(x)}{\delta n} = \sum_{m=0}^{n} \pm 1$$

It is not hard to see that 33 is an even integer if n is odd and an odd integer if n is even. Hence 33 does not approach a limit. Hence by the previous discussion, $f(x)$ does not have a derivative.

It might seem that the cause for the non-differentiability of van der Waerden's function is the presence of cusps on the functions $f_n(x)$ which might tend to produce concentrations of cusps on $f(x)$. This is not the true cause, however, for Weierstrass' function

$$(34) \qquad f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$$

where a is an odd integer and $ab > 1$, is not differentiable anywhere, and differs from van der Waerden's function chiefly in that it uses sinusoidal waves instead of sawtooth waves. On the other hand,

$$(35) \qquad f(x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!}$$

has cusps wherever any of the functions $f_n(x)$ has a cusp, but is differentiable at all other points.

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A STATISTICAL PROBLEM IN ADVERTISING

C. D. Smith

Profit from advertising is of interest to any business operation. A statistical approach to the problem may be based on the assumption that advertising expense is justified only when the increase in net profit from sales is greater than advertising cost. The many variables in our economic system make it very difficult to treat separate causes of a given event. H. V. Roberts[1] established a case in point by computing correlations for a set of factors that should influence advertising. He obtained relatively small values for simple correlations and partial correlations. In conclusion he points out that one may expect very little information from correlation methods. Other studies have been published which seek to compare advertising techniques by use of multiple regression coefficients. In a recent report by Stephan[2], 'History of the Uses of Modern Sampling Procedures', has no reference to use of sample design in advertising research. The purpose of this paper is to give a sample approach tc the matter of advertising cost.

To test the value of advertising by measuring gain in a controled experiment one may proceed as follows. Select area A to include a sufficiently large number of towns. For each town in A select a sample of prospective stores which ordinarily stock a product of type P. For each town select the sample as follows.

1. Select as center block the one with the largest number of stores of the required type. Choose one at random for the sample. From the zone of stores adjacent to the center block choose one at random of required type.
2. Continue the selection from zones adjacent to the preceding zone until the outside area has no block with more than one store of required type. Select one at random from this outside area.
3. Proceed in like manner from town to town until the sample contains at least one hundred stores.

Begin the experiment by placing product P in each sample store and list the name without comment in the usual storewide advertising space. Calculate net profit from sale of P over a given period. Now begin special advertisements featuring P and sell for a given period. Increase the special advertising cost at successive periods. Let the cumulative cost of advertising at the end of a given period be X_i, the corresponding net profit be Z_i, and the advertising value V_i be given by the formula $V_i = Z_i - X_i$. In due time the value of V will

1. The Journal of Business, University of Chicago. Vol. 22, No. 3, July 1947.
2. Journal, American Statistical Association. Vol. 43, No. 241, March 1948.

show a decrease compared to the previous period. This indicates the point where the greatest profit has been attained. Although we cannot say that V is produced solely by the increase in X, we can say that X should not increase beyond the point which maximizes V.

Note in conclusion that a basic value of V was esdablished before values of X were assigned. In this way causes apart from X have been removed to this extent from the final value of V. If some other factor is believed to increaee with X, that factor should be checked over the sample of stores for a sufficient period. We may say that the X which gives the maximum value of V is a signal beyond which one should not spend.

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A COMPARISON OF THE UNITED STATES RULE WITH THE MERCHANT'S RULE FOR COMPUTING THE MATURITY VALUE OF A NOTE ON WHICH PARTIAL PAYMENTS HAVE BEEN MADE

Joseph Barnett, Jr.

FOREWORD

When partial payments are made on a note before the maturity date there are at least two ways to compute the amount due the holder of the note when it is due. These are designated as the United States Rule and the Merchant's Rule. The rule to be used usually depends upon an agreement between the parties concerned at the time the note is given. However, the United States Rule is based on a decision of the United States Supreme Court to the effect that it is not legal to charge compound interest on a debt. It is the purpose of this paper to show that for notes having large face values, the difference between the maturity values computed by the two methods may be large.

The United States Rule may be stated as follows:

> Simple interest is computed on the note from the time it was given to the time of the first payment by the use of compound time. If the payment is equal to or greater than the interest, it is subtracted from the sum of the face of the note and the interest. Interest is computed for the periods between successive payments on the remainders resulting after subtracting each payment from the sum of principal and interest due at the time it was made; and the process continued until the time of maturity of the note. The result, so obtained, is the maturity value of the note.

But if the payment at any time is less than the interest, the interest is not added to the principal nor the payments subtracted until the time at which the sum of the payments not having been subtracted is

greater than or at least equal to the sum of the interests not having been added. Then the sum of the interests is added to the principal and the sum of the payments subtracted. Then interest is computed on the remainder as aforesaid.

In order to compare the two methods, I shall alter the statement of the Merchant's Rule so as to show more clearly the difference of the two methods, and still obtain the same results by its use as those obtained by using it in its usual form. The altered Merchant's Rule may be stated as follows:

Compute the simple interest using compound time on the face of the note from the time it was given to the time of the first payment. From the face of the note subtract the first payment. Compute the interests for the time between successive payments on the remainders of the face of the note after subtracting the payments when they were made. Continue this process until the time of maturity of the note. Then the sum of the interests having accrued at that time is added to the residue of the face of the note. The sum is the maturity value of the note.

Let it be observed that in the case of the Merchant's Rule no interest is added before all the payments have been made, and that in this fact lies the difference in the results obtained by the two methods.

Suppose we use the Merchant's Rule in a problem in which p = the face of the note; r = the rate of interest; $p_1, p_2, p_3, \ldots p_n$ = the 1st, 2nd, 3rd, ... and nth payments, respectively; $i_1, i_2, i_3, \ldots i_n$ = the interest computed for the 1st, 2nd, 3rd, ... and nth periods, respectively; $t_1, t_2, t_3, \ldots, t_n$ = the time between payments, respectively. Hence the maturity of the note is

$$(1) \qquad A = P - p_1 - p_2 - p_3 - \ldots - p_n + i_1 + i_2 + i_3 + \ldots I_n.$$

However, if we use the United States Rule, and let $I_1, I_2, I_3, \ldots I_n$ be the interest computed for the 1st, 2nd, 3rd, ... nth periods respectively, the maturity value will be

$$(2) \qquad A' = P - p_1 - p_2 - p_3 - \ldots - p_n + I_1 + I_2 + I_3 + \ldots + I_n.$$

If equation (1) is subtracted from equation (2), we have an expression for the difference in the maturity values by the two methods:

$$(3) \qquad A' - A = I_1 - i_1 + I_2 - i_2 + I_3 - i_3 + \ldots + I_n - i_n.$$

And since $I_1 = Prt_1$, $i_1 = Prt_1$; $I_2 = (P + I_1 - p_1)rt_2$, $i_2 = (P - p_1)rt_2$;

$$I_3 = (P + I_1 + I_2 - p_1 - p_2)rt_3$$

$$= (P - p_1 - p_2)rt_3 + (I_1 + I_2)rt_3,$$

$$i_3 = (P - p_1 - p_2)rt_3; \ \ldots \ ;$$

$$I_n = (P + I_1 + I_2 + \ldots + I_{n-1} - p_1 - p_2 - \ldots - p_{n-1})rt_n$$

$$= (P - p_1 - p_2 - \ldots p_{n-1})rt_n + (I_1 + I_2 + \ldots + I_{n-1})rt_n,$$

$$i_n = (P - p_1 - p_2 - \ldots p_{n-1})rt_n,$$

$$A' - A = I_1 rt_2 + (I_1 + I_2)rt_3 + (I_1 + I_2 + I_3)rt_4 + \ldots + (I_1 + I_2 + \ldots I_{n-1})rt_n. \tag{4}$$

In order to take care of the case in which the interest is greater than the payment, let the rth interest be greater than the rth payment. We compute the interest on the remainder after the $(r - 1)$th payment has been subtracted up to the time at which the sum of the payments not having been subtracted equals or exceeds this interest. Then compute the interest on the balance as aforesaid. The maturity value in this case is

$$A' = P + I_1 + I_2 + \ldots I_{r-1} + \ldots I_{n-m+1} - p_1 - p_2 - \ldots - p_n$$

in which m is the number of payments after the $(r - 1)$th up to the time at which the sum of the payments not having been subtracted equals or exceeds the interest not having been added.

For the trivial case in which r is zero, t is zero, or both are zero, $A' = A$; for r and t equal to any positive value, however small, P can be made sufficiently great to make $A' - A$ as large as we please. If, for example, $r = 0.1\%$, $t_1 = 1$ day, and $t_2 = 2$ days, and $P =$ \$360x180x1,000,000,000, since $I_1 rt_2 = Pr^2 t_1 t_2$, the first part of the difference (4)

$$A' - A = (\$360)(180)\times 1{,}000{,}000{,}000(.001)^2 \frac{1}{360}\frac{1}{180}$$

or $A' - A = \$1000$. However, it should be remarked that in the cases of practical importance, the differences in the maturity values computed by the two methods are insignificant.

Oklahoma A. and M. College

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Number Theory and its History, by Oystein Ore, New York, McGraw-Hill (1948) 370 pages, $4.50

This book deals with the principal ideas of elementary number theory in the order of their historical development, beginning with the counting processes of savages, and ending with Gauss' theory of the ruler and compass construction of the regular heptakaidecagon. As history, it is far more interesting than the usual elementary historical text, since the author explains the mathematical ideas whose development he is tracing in sufficient detail so that the reader can grasp their significance. Considered as a text on elementary number theory, the space given to historical treatment precludes the inclusion of much material usually considered essential. For example, there is no account of the quadratic reciprocity law, continued fractions, numerical functions and their inversion, quadratic forms, Bernouilli numbers.

These ommissions are unfortunate of course, but deliberate. Professor Ore has adopted a rather novel viewpoint for a mathematical author; he has assumed both the existence of other accessible books on his subject, and the existence of readers of sufficient intelligence and maturity to refer to these books if they should desire more information.

The strongest feature of Ore's 'Number Theory' is that it has something fresh to say about almost every subject it treats. These subjects range from historical topics such as recent discoveries in Chaldean trigonometry to Axel Thue's beautiful proof of Fermat's theorem on the representation of primes as sums of two squares. There is a sketch of the basic ideas of lattice theory and up-to-date information on methods of factoring large numbers. The treatment of congruences is very full and clear.

The book should be of interest to amateurs in the theory of numbers, teachers and prospective teachers of mathematics. Although it is not a formal text book, it could very well be used for a 'survey' course

for non-mathematicians or for a teacher's training course, particularly if the instructor would supplement it slightly.

The book is lucidly and entertainingly written. Professor Ore has a quiet humor which is in refreshing contrast to the labored facetiousness which many recent authors of elementary surveys have felt bound to impose upon themselves, and upon their readers.

Morgan Ward

Solid Geometry. By J. S. Frame, McGraw-Hill Book Co., 1948, 19 plus 339 pages. $3.50.

The phases of solid geometry which are primarily emphasized in this text are solid mensuration, and the drawing of figures. The concept of a proof, though included in the text, is relegated to a position of minor importance. In fact many of the proofs are left as exercises for the student, and thereby could be worked either by the student, or the teacher, ar omitted.

The book is divided into four parts of ten chapters each. The headings of the various parts are Linear and Angular Measurement in Space, Solid Mensuration, The Sphere and Solids of Revolution, and Projections and Maps. The material covered in the first twenty five chapters could be said to contain the material covered in the customary short course in this subject. The remaining chapters could be used for special reports, or for additional content for a longer course in Solid Geometry.

In addition to the traditional solid geometry which is included in this text, one finds an introduction to the study of vectors, the introduction of the cosine of an angle as a projection factor, chapters on the celestial and terrestial sphere, and a few properties of the conic sections which are developed synthetically when the plane sections of cones and cylinders are studied. Much space is devoted to and many rules are given for the drawing of the projections of space figures.

At the end of each chapter is a list of oral exercises which should prove stimulating to the superior student. There are also lists of problems for written work, most of which involve either numerical computation or the drawing of a specified figure.

Throughout the text many situations are described, which involve statements which to the sophisticated reader demand proof. Also, certain terminology is occasionally used, such as additive measure, or linear space, which might cause difficulty to the immature student.

For those who want a text in solid geometry with a minimum of proof, and a great deal of numerical computation and drawing of

figures, this book should prove to be a satisfactory text. For those who expect a book on Solid Geometry to emphasize postulates, axioms, and synthetic proofs, which traditionally occur in a course on this subject, this work would probably not meet with their approval.

R. G. Sanger

Solid Analytic Geometry. By Adrian Albert, McGraw-Hill Co., 9 plus 162 pages. $3.00.

In this book an attempt is made to develop some of the concepts of Solid Analytic Geometry by means of vectors and matrix theory. Thus, utilizing the concept of a vector, certain formulas involving planes and lines can be developed in a neat manner, and utilizing matrices, the problems involving transformation of coordinates can be related to certain phases of matrix theory.

The text is very tersely written, and additional illustrative examples and figures could be used to good advantage in many places. There is at least one place where an algebraic equation is broken at the end of a page, and at least one place where a short matrix equation is broken at the end of the line. Such things, together with the inevitable misprints which occur in the first printing of any work, do not make the book easy reading.

There are nine chapters in the book, the first five of which include material on planes, lines, and quadric surfaces in standard position; material traditionally included in a first course in this subject. The sixth chapter is on matrix theory, and the seventh deals with the problem of rotation of axis and classification of quadric surfaces. The last two chapters deal with spherical coordinates and the elements of projective geometry.

The main objection to this work is not that it is concisely written, but that much is omitted which one would expect to appear in such a book. Some of these omissions are noted in the following paragraphs.

Nowhere in the preface or in the text is there any indication what mathematical background a person should have before attempting to read this book with understanding.

In the first two sections vectors and the ideas of linear independence and dependence are introduced, but practically no indication is made of the geometric or physical significance of a vector, and no algebraic criterion is given or implied whereby a student could tell whether or not two vectors were linearly dependent. In the next section a dot or inner product is introduced and an expression is given for the cosine of the angle between two vectors, yet no geometric interpretation of this expression is made until some sections later. Nowhere is the vector cross product introduced, nor are any geometric properties given which can be concisely formulated utilizing this concept. Also, in the development of equations of tangent planes

to surfaces, no use is made of vectors.

Most of the exercises in the text are very formal in nature, in general, either consisting of substituting in formulae already developed, or of working out a parallel theory for a slightly different case. There are no problems involving the concept of a locus, and none involving a geometrical analysis of a particular situation.

Figures illustrating the geometric significance of the theory are rare. There are no figures in the chapters headed Surfaces and Curves, Spheres, Rotations of Axis and Applications, and Elements of Projective Geometry. There is only one figure in the chapter dealing with planes.

When equations of lines in symmetric form are given, the number zero appears in some denominators, and, though there immediately follow equivalent relations which are stated as valid (without too much explanation), the effect on the average student of seeing such things in print leaves much to be desired.

In defining the angle between two planes, it is stated that such an angle is not uniquely defined if either plane passes through the origin. This is consistent with the definitions in the text, but is not with the geometrical situation, yet no attempt is made to clarify this point.

Nowhere is the concept of determinant used in the main portion of the text, though they are discussed in the latter part of the sixth chapter. Thus, the concise determinantal formulae for the equation of a plane through three non-collinear points, or spheres through four non-coplanar points are omitted.

In the study of the reduction of the general equation of a quadric surface to canonical form, matrices are used and the problem is associated with that of reduction of quadratic forms to canonical types. Nowhere is the rank of a matrix used, or are the invariants, associated with the equation of the quadric under translations or rotations, with the exception of the roots of the characteristic equation, mentioned. This is to be regretted, since the concept of rank of a matrix can be advantageously used both in questions of linear dependence and in the problem of classifying the quadric surfaces in a systematic manner.

In the chapter on spherical coordinates, the traditional names for the coordinates, ρ, θ, ϕ are disgarded. No applications of a practical nature are made, though one or two are vaguely suggested. Similarly, in the chapter on Projective Geometry, projective coordinates are associated with vector spaces, and projective transformations with matrices, but there is no indication as to what a projective geometry is, or what it might be used for.

As the book is tersely written, is lacking in illustrative examples, and also lacks figures to aid those who are visually minded, it is doubtful if the average student would gain much by trying to read this book independently.

In the study of geometry, certain tools, such as vectors, matrices, and derivatives, may be used in the development of the geometric theory. In this work, there are spots where the study of the properties of the tools used are definitely emphasized, the geometric aspects of the situation being relegated to a secondary position.

R. G. Sanger

Comments on and additions to H. V. Craig's paper 'On Extensors and the Lagrangian Equations of Motion', Vol. XXII No. 5, March-April of the Mathematics Magazine. By C. W. Horton.

1. *Introduction.* In a recent paper H. V. Craig has shown how the Lagrangian equations of motion may be derived from a simple extensor equation relating the primary extensors associated with kinetic and and potential energies. His development is confined to conservative forces which may be represented by a potential function. Although this covers the majority of cases, it is of interest to remove this restriction and to inquire if an equally simple basis may be found for the equations of motion for non-conservative forces.

2. *The excovariant force function.* Suppose that the forces acting on the particles in the system may be represented by a covariant tensor Q_a. The sense of the components of Q_a are such that they are positive when the force is directed along positive x^a. Consider the set of quantities $Q_{\alpha a}(\alpha = 0, 1)$ defined by

$$\begin{aligned} Q_{0a} &= Q_a \\ Q_{1a} &= 0 \end{aligned} \tag{1}$$

and inquire whether or not they constitute an extensor of range 1. The extensor transformation law

$$\overline{Q}_{\rho r} = Q_{\alpha a} X^{\alpha a}_{\rho r} \tag{2}$$

gives, for the case $\rho = 0$,

$$\overline{Q}_{0r} = Q_{0a} X^{0a}_{0r} + Q_{1a} X^{1a}_{0r} \tag{3}$$

By equation (1) the Q_{1a} are zero so equation (3) reduces to

$$\overline{Q}_{0r} = Q_{0a} X^{0a}_{0r} = Q_a X^a_r = \overline{Q}_r \tag{4}$$

Since Q_a is a covariant tensor of order 1.

When $\rho = 1$, equation (2) becomes

$$\overline{Q}_{1r} = Q_{0a}X^{0a}_{1r} + Q_{1a}X^{1a}_{1r}. \tag{5}$$

Now X super 0a inf 1r vanishes since x^{aa} is a function of only those $x^{\rho r}$ for which $\rho \leq a$. The Q_{1a} vanish by equation (1), and hence, $\overline{Q}_{1r} = 0$. Thus it is seen that the Q_{aa} constitute an excovariant extensor of range 1.

3. *The heuristic formulation.* The heuristic formulation of Dr. Craig may now be modified to read:

(*a*) The important functions are the kinetic energy $T(x,x')$, which is homogeneous of degree two in the x''s, and the excovariant force function Q_{aa} of range 1. It is assumed that a linear relation between the primary extensors associated with T and with the force extensor Q_{aa} holds along a trajectory.

(*b*) The increase of T along a trajectory is equal to $Q_a x'^a$. This has the simple physical interpretation that the increase in the kinetic energy is equal to the work done by the forces.

(*c*) The extensor component of functional order two, namely, T_{0a} must be present.

These assumptions lead to an extensor equation of the form

$$T_{aa} + AT_{;aa} + BQ_{aa} = 0. \tag{6}$$

The rank of equations obtained when $a = 1$ will be satisfied identically if $A = -1$ since $T_{1a} = T_{;1a}$. When $a = 0$, one has, since $T_{0a} = T_{;1a}'$,

$$T_{;1a}' - T_{;0a} + BQ_{0a} = 0. \tag{7}$$

When this equation is multiplied by x'^a, it can be reduced to

$$T' + BQ_{0a}x'^a = 0.$$

The details of the analysis are given in Dr. Craig's paper. Assumption (*b*) requires that $B = -1$, and, consequently, equation (7) becomes

$$T_{;1a}' - T_{;0a} = Q_{0a} = Q_a \tag{8}$$

which is identical to the form of the Lagrangian equations given by Whittaker[2].

BIBLIOGRAPHY

1. H. V. Craig, *On Extensors and the Lagrangian Equations of Motion*
2. E. T. Whittaker, *Analytical Dynamics*, p. 37, Dover Publications, Inc., New York, 1944.

The University of Texas

PROBLEMS AND QUESTIONS

Edited by

C. G. Jaeger, H. J. Hamilton and Elmer Tolsted

This department will submit to its readers, for solution, problems which seem to be new, and subject-matter questions of all sorts for readers to answer or discuss, questions that may arise in study, research or in extra-academic applications.

Contributions will be published with or without the proposer's signature, according to the author's instructions.

Although no solutions or answers will normally be published with the offerings, they should be sent to the editors when known.

Send all proposals for this department to the Department of Mathematics, Pomona College, Claremont, California. Contributions must be typed and figures drawn in india ink.

SOLUTIONS

In addition to those listed in the March-April 1949 issue, the following also solved Problem No. 27: *G. W. Courter*, Baton Rouge, La.; *W. R. Talbot*, Jefferson City, Mo.; *G. B. Knight*, Oak Ridge, Tenn.

No. 24. Proposed by *C. N. Mills*, Normal, Illinois.

Given the quadratic form $x^2 - x + N$ where $N = 2, 3, 5, 11, 17, 41$. When $x = 1, 2, \ldots (N - 1)$ each of the resulting numbers is prime. Are there other values of N?

D. H. Lehmer has shown that there is at most one other value of N and if it exists it exceeds 1,250,000,000. (*Sphinx*, 1936, pp.212-214). This is referred to in Coxeter's edition of *'Rouse Ball' Mathematical Recreations and Essays* p. 62.

Leo Moser, Winnipeg, Canada.

No. 29. Proposed by *Norman Anning*, Ann Arbor, Michigan.

Given that n is any positive integer greater than 1, show that the curve $\dfrac{1+y}{1-y} = \left(\dfrac{1+x}{1-x}\right)^n$ has three and only three points of inflexion.

Solution by *W. R. Talbot*, Jefferson City, Missouri.

The given equation $f(x,y) = 0$ may be written explicitly in y as

$$y = \frac{(1+x)^n - (1-x)^n}{(1+x)^n + (1-x)^n}.$$

In this form it is readily seen that the curve has symmetry with respect to $(0,0)$. It is sufficient to consider only $x > 0$. In terms of logarithms, the given curve $f(x,y) = 0$ may be written as

$$\ln \frac{1+y}{1-y} = n \ln \frac{1+x}{1-x},$$

so that

$$y' = n\,\frac{1-y^2}{1-x^2}$$

and

$$y'' = -2y'(ny - x).$$

If $y'' = 0$, then $y' = 0$ and $ny - x = 0$. If $y' = 0$, then $y = \pm 1$ and from $f = 0$, $x = \pm 1$. Consider values of x slightly less and greater than 1. If n is even, $y < 1$ for these values of x; so that $(1,1)$ is a bend point. If n is odd, $y < 1$ when $0 < x < 1$, but $y > 1$ when $x > 1$; so that $(1,1)$ is an inflexion point. By virtue of symmetry $(-1,-1)$ is an inflexion point if n is odd.

It remains to be shown that $ny - x = 0$ has precisely three roots if n is even and only one root if n is odd. Let $g(x)$ denote the result of substituting the explicit value of y above into $ny - x$. Then $g(x) = 0$ is

$$(1+x)^n(n-x) - (1-x)^n(n+x) = 0.$$

Since $g(0) = 0$, $(0,0)$ is an inflection point.

Let n be odd. Then for $0 < x \leq n$, $g(x) > 0$. Dividing $g(x)$ by n and allowing n to become exceedingly large is equivalent to evaluating $(1+x)^n - (1-x)^n$, which is positive. Then if n is odd, $ny - x = 0$ has no root other than $(0,0)$.

Let n be even. We find $g(1) > 0$, $g(n) < 0$ indicating a root between 1 and n, and from symmetry, one between -1 and $-n$. If $x > n$, $g(x) < 0$ because both $(1+x)^n(n-x)$ and $-(1-x)^n(n+x)$ are negative. Then, there is no root beyond $x = n$ (or $-n$). Whether n is odd or even, there are precisely three points of inflexion.

Solved also by *Leo Moser*, Winnipeg, Canada.

No. 30. Proposed by *Victor Thebault*, Tennie, Sarthe, France.

Given a circle (O), of center O, tangent to two rays Ax and Ay. A variable tangent meets Ax at B and Ay at C. Show that each of two sides of the triangle which has the orthocenters of triangles AOB, BOC, COA as vertices pass through fixed points, and that the third side has a conic as its envelope.

Solution by *O. J. Ramler*, Catholic University.

Let the fixed circle (9) be tangent to BC, CA, AB at A', B', C',

respectively, and let the orthocenters of triangles AOB, BOC, COA be H_c, H_a, H_b respectively. Then H_b and H_c move on the fixed lines $B'O$ and $C'O$ as side BC envelopes the circle (O). Moreover H_bC and H_cB are each perpendicular to the fixed line OA. Since B and C are corresponding points in projectively related ranges of points on Ax and Ay respectively, it follows that H_b and H_c are corresponding points in projectively related ranges of points on the fixed lines OB' and OC'. Hence H_bH_c envelopes a conic which is a hyperbola having OB' and OC' as asymptotes because the axis of homology for the projectively related ranges on OB' and OC' is the line at infinity. Now H_b and H_c uniquely determine H_a. Hence H_aH_b cuts Ay in points Y projectively related to H_b. Now when triangle ABC degenerates so that C is at B', B will be at A and H_b will be at B'. Then Y will also be at B'. Hence since B' is self-corresponding, the lines YH_b will not envelope a proper conic; they will pass through a fixed point, *i.e.* $H\ H$ and similarly H_cH_a pass through fixed points. When B is at infinity on Ax, $A'OC'$ is a diameter of circle (O), H_b will be at O, and H_a will lie on OC'. Hence the fixed point through which H_aH_b passes lies on OC'. When C is the point where $C'O$ cuts, Ay, H_a and H_b each lie on Ay. Hence the fixed point is C'. Similarly H_aH_b passes through B'.

No. 32. Proposed by *Victor Thebault*, Tennie, Sarthe, France.

Find all the four-place numbexs $abcd$ and $aecd$ which are perfect squares.

Solution by *W. R. Talbot*, Jefferson City, Missouri.

Let $abcd = m^2$ and $aecd = n^2$. From $m^2 - n^2 = 100(b - e)$, it follows that both $m + n$ and $m - n$ are even. If m^2 and n^2 are different, then $64 < m + n < 198$. If $b - e = 9$, the maximum value of $m - n$ is given when $a = 1$; thus $m - n$ is an even number less than 11. If $m - n = 2,4,$ or 6, $m + n$ may be 100 or 150. If $m - n = 8$, $m + n = 100$ while $m - n = 10$ does not lead to any solutions. Of the possible values of (m,n), only $m - n = 6$ and $m + n = 150$ is unacceptable, and the solutions are

(51,49)	2601, 2401	(52,48)	2704, 2304	(53,47)	2809, 2209
(76,74)	5776, 5476	(77,73)	5929, 5329	(54,46)	2916, 2116

Solved also by *Leo Moser*, Winnipeg, Canada.

No. 34. Proposed by *Victor Thebault*, Tennie, Sarthe, France.

Find a number of four digits such that its square ends with the

same four digits in the same order. Show that its cube and its fourth power have the same property.

Solution by *Leo Moser*, Winnipeg, Canada.

All powers of 9376 end in 9376.

The theory of automorphic' numbers is well known. It is discussed in some detail in Kraitchik, *Mathematical Recreations* pp. 77-78 where it is shown that the following two numbers are automorphic.

3,740,081,787,109,376
6,259,918,212,890,625

The fact that the second of these ends in 0625 disqualifies it as a solution to the problem as stated.

No. 35. Proposed by *Victor Thebault*, Tennie, Sarthe, France.

A plane P divides the volume of a sphere into two parts V and V' and determines two spherical segments of areas S and S'. If it is known that $\frac{S}{S'} = k$, calculate the ratio $\frac{V}{V'}$ in terms of k.

Solution by *Leo Moser*, Winnipeg, Canada.

Take the radius of the sphere to be $1 + k$ and cut it by a plane a distance k from the top. The ratio of the areas of the segments will then be $k:1$, and a simple calculation using the well known formula for the volume of a spherical cap, volume $= \frac{h}{6}(3a^2 + h^2)$, yields $\frac{V}{V'} = k^2\frac{(3 + k)}{(3k + 1)}$.

PROPOSALS

37. Proposed by *Leo Moser*, Winnipeg, Canada.

I would like to propose the following problem. (Suggested by game of Russian Billiards).

Given 5 points in or on a 2×1 rectangle. Show that the smallest distance determined between any 2 of them is $\leq 2\sqrt{2 - \sqrt{3}}$ and that this is the largest number for which the result is true.

38. Proposed by *Leo Moser*, Winnipeg, Canada.

Show that 5 or more great circles on a sphere, no 3 of which are concurrent, determine at least one spherical polygon having 5 or more sides.

39. Proposed by *Leo Moser*, Winnipeg, Canada.

Let $A_{ij} = \dfrac{(i + 1)^{j+1} - 1}{(i + 1)!}$, $i \quad 1, 2 \ldots n$, $j \quad 1, 2 \ldots n$.

Show that the nth order determinant $|A_{ij}| = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n+1}$

40. Proposed by *Pedro A. Piza*, San Juan, Puerto Rico.

Let x be a positive integer and $S_n = 1 + 2^n + 3^n + 4^n + \ldots + x^n$. Prove the following Pythagorean relations:

$$(48S_7 + 160S_9 + 48S_{11} + 1)^2 = (8S_3 + 24S_5)^2 + [(4s_3 + 12S_5)^2 - 1]^2.$$

$$(64S_9 + 448S_{11} + 448S_{13} + 64S_{15} + 25)^2 =$$

$$(160S_5 + 160S_7)^2 + [(16S_5 + 16S_7)^2 - 25]^2.$$

41. Question by *Raymond L. Krueger*, Wittenberg College.

I wonder if you can tell me anything about the following problem. It was sent to me by a former student and we do not know the original source nor can we seem to interpret it correctly.

3 children: one has lived a diminished evenly even number of years, another a number also diminished, but evenly uneven, while a third, an augmented number unevenly even. What are the ages of the children?

TEACHING OF MATHEMATICS

Edited by **Joseph Seidlin and C. N. Shuster**

VARIETY OF MATHEMATICAL EXPERIENCES*

Harold E. Bowie

American International College, being the only co-educational college of Arts and Sciences in Springfield, considers its function to be that of a Peoples' college. As to mathematics, we require a year of algebra and a year of plane geometry for entrance. Some candidates with prospective ability are admitted without these courses and complete them later in our summer session or elsewhere.

Those who plan to major in mathematics or one of the natural sciences are expected to come to us with at least a third year of high school mathematics. We give a three semester hour course in intermediate algebra for those who enter with the minimum requirement.

As we do not require any mathematics on a college level, the two courses required for entrance are terminal for the majority of our students.

In the double track plan it is very important, I think, not to lose sight of the individual. An early determination as to whether one will make a good mathematician is difficult in many cases, both for him and his advisers. We have had a number of cases of students who came to us with little or no secondary school mathematics who majored successfully in the subject by doing summer work or staying on for a fifth year.

Without the rather unusual program which A. I. C. has set up to take care of such cases, some of our good mathematics people would be eliminated. Generally, students of college age dislike to take these courses with the younger pupils in the high schools. In using the double track or any other plan, provision should always be made for switches along the way.

A little fanning of the flame is a good thing for those who show natural interest in our subject. Many students have been influenced to continue their mathematics by the enthusiasm of some teacher. We are in a field for which we do not need to apologize and should give encouragement to those who show interest in it. The world needs them.

Often the work in hand may be used as a point of departure for digression into the uses, history, and scope of the subject. For example, I never discuss the ellipse withour saying a little about Kepler, Newton, and Einstein. Not everything I say is understood fully, even by myself, but many a good mathematician has started his career under the influence of such inspiration.

Many of my students say that they didn't learn algebra until they used it in the calculus. The better students overcome this weakness in their background without too much difficulty. The mediocre ones

*Paper read at the spring meeting of the Conn. Valley Section of the Assoc. of Teachers of Math. in New England at Suffield, Connecticut, April 23, 1949.

have a hard time. The poor student is sunk. These pupils readily admit that one reason for their lack of preparation was that they didn't study.

Too hard or too easy standards of marking may cause students to stop studying. No student should be given credit in a course until he attains a certain minimum of achievement. I have seen many do a good job on a course with a second attempt. One pupil who failed in every period of plane geometry came back the next year with all A's, and is now an excellent teacher of mathematics.

Lack of variety of experience in high school and early college courses is a source of some difficulties. In practice with the four fundamental operations, many situations with respect to signs and symbols should be included. Zero is a number which confuses most students, even when they have reached quite advanced work. This is not strange, as zero is involved in the only fundamental operation which is undefined.

Telling the student that division by zero is undefined is not sufficient. Sets of exercises should include those involving zero terms, factors, numerators, and denominators, so that he may learn to readily recognize the impossible situation. He should have experiences with such expressions as $\frac{1}{x}$ when $x = 0$, $\frac{1}{(x-2)}$ when $x = 2$, and $\frac{(x^2 - y^2)}{(x - y)}$ when $x = y$. these expressions have no meaning in an elementary sense under the conditions given.

Upon reading the last paragraph aloud, I found myself saying 'one over x' for 'one divided by x'. During my first years of teaching, the dean and former head of the department of mathematics at my Alma Mater visited our high school on a student recruiting mission. He and the professor with him spent the evening at my home. During the conversation, which naturally turned to talking shop, he pointed out emphatically that one should never say 'over' for 'divided by'. Being guilty of this fault myself, I was somewhat embarassed. As a teacher I have found that many such early habits have to be overcome.

I still like 'over' for 'divided by' and use it in conversation with those of sufficient mathematical maturity not to be confused. It is shorter. These short cuts are dangerous with beginning students who have not become sufficiently familiar with the symbols involved.

It is not well, however, for us as teachers to accept ideas because they have been printed in some book or stated by some authority.

As a part of my graduate work, I was required to visit a class in a large city high school. The work in hand was the solution of linear equations. The teacher was having a bad time of it. She was religiously adhering to an artificial, involved scheme that had been presented in a course in the Teaching of Mathematics the preceding summer. Too much time used with a few pupils caused the rest of the class to lose interest, and the period ended in confusion with little done. She was quite discouraged and asked me what I thought she could

do. I advised her to abandon the idea and return to well-tried methods. The scheme was doubtful with any class and impossible with a large class.

Speaking of tried methods reminds me that one of our campus weekly reporters once asked me what I was going to talk about. Not knowing at the time, and wishing to give him some title, I said the first thing which occurred to me, which was 'The Double Track Plan'. It appeared in print later as 'The Durable Track Plan'. Now I am wondering whether this would make a good subject for study.

One of the hurdles which slows the calculus student is the handling of complex fractions. I seem to remember having read somewhere in educational literature that practice with complex fractions was time wasted. Such is not the case. There are many problems in calculus, for example, which involve fractions of various levels of complexity. Thus the differentiation of tan $\frac{1}{x}$ requires simplification of a fraction of medium difficulty, while the differentiation of tan $\frac{x}{(x^3 + 1)}$ presents a more challenging situation.

The average calculus student needs to be skillful with these fractions so as not to be diverted from the many new concepts to be learned. Practice with various forms of complex fractions should be included in the algebra courses and should be repeated at intervals throughout the mathematics program.

These and later considerations would underline the fact that there is a minimum of training that is desirable for all mathematics teachers. This matter has been given serious attention by the Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics. Their conclusions are found in their report on the Place of Mathematics in Secondary Education in the Fifteenth Yearbook, issued in 1940[1].

These recommended requirements in mathematics include a year of calculus, a brief introduction to projective and non-Euclidian geometry, using synthetic methods, advanced algebra, and history of mathematics. Recommended requirements in natural science and education are also listed as well as further desirable but less critical needs. Some of us taught high school mathematics for a time with fair results without all of these requirements. There is no doubt, however, that we did better work after getting a solid background.

Experiences with the handling of expressions involving a fixed constant base and literal exponents are necessary. Students have trouble multiplying in such cases as $2^n(2^{n+1} - 1)$. Factoring and division with such quantities is even more perplexing. Often students who have this trouble can handle similar operations with a literal base and numerical exponents successfully. Considerable practice

[1] Fifteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1940) pp. 201-202.

with fixed constant base and literal exponents is incidental to the usual sets of exercises in mathematical induction. For example, prove by mathematical induction that $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + n \cdot 2^n = 2 + (n - 1)2^{n+1}$ where n denotes a positive integer.

Mathematical induction is hard to teach to high school students or college freshmen. Previously gained familiarity with the methods of simplification involved still leave enough concepts hard to grasp. The understanding of mathematical induction itself requires a variety of experiences. The usual set of problems found in most college algebras involve a proof requiring the addition of something to both sides of an equation and are always true. Problems where some other operation is performed on both sides or which are not true may be provided by the teacher.

C. S. Carlson gives six exercises that are true for $n = 1,2,3,4$, but fail for $n = 5$, in the National Mathematics Magazine for October, 1944[2].

Ability to insert parentheses where they will be advantageous and to remove them when they have served their purpose is indispensable to the student. It enables him to free his mind from the details of simplification and to concentrate on basic principles until a later or final step. It will occur to some that here is the place to learn about parentheses. That is, to wait until they are needed.

It is true that skill will be maintained and increased incidentally at this point. However, the game must not be delayed very much or the whole situation becomes disagreeable to the student. For interest to be maintained while a student is integrating $\sin^4\theta$, for example, results must be gotten without being held up for long by side issues.

Simplification of complex expressions with radicals, and fractional and negative exponents involved are good preparation for later work. Throwing an expression into a good form for differentiation or integration often requires a change from radical to exponential form or vice versa, or the changing of an expression from numerator to denominator or the reverse.

Some of the properties of proportions studied in elementary algebra and geometry and then abandoned for so long that they are almost entirely forgotten may become incidental to the work of later courses. This lapse of time between the last mathematics course taken in high school and the beginning of college mathematics and science seems to be one of the unsolved problems of curriculum making. The lapse is often one or two years. Another problem of the curriculum is that many who follow mathematics do not get solid geometry because it is not required in the high school and not offered in college. These problems arise because many who will go into mathematics and science do not know it when they are in secondary school.

[2]C. S. Carlson, 'Note on the Teaching of Mathematical Induction', National Mathematics Magazine, Vol. 19, 1944.

The importance of emphasizing functional relationships has been deservedly emphasized in the literature. Practice with the symbolism as soon as it can be understood is important and should be given renewed attention from time to time. The student of college mathematics needs to be acquainted with the shades of meaning attached to $f(x)$, $f(a)$, $f(2)$, $\phi(x)$, $F(x)$, etc. He should recognize quickly whether $f(x)$ is being used to represent a particular or general function by the context. Early familiarity with these symbols will free his mind for reasoning in which their significance is incidental.

Some beginning calculus students will differentiate $x^2 + a^2$ to obtain $2x + 2a$ where the context should make it clear that a is a constant. They ignore the fact that usually, although not always, the first letters of the alphabet are used conventionally to denote constants. The difference between an arbitrary constant and a variable is a very subtle thing and requires good explanations by the teacher and hard thinking by the student.

Exercises in which explanations written out in words predominate over juggling of symbols are good to promote reasoning. It is not uncommon for an entire class to differentiate a number of expressions correctly and to balk unanimously when asked to show $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$.

It is well known that many of the troubles of mathematics students with relatively difficult theory and problems comes from lack of reading ability. The first requirement in the understanding of a new bit of theory or a problem is a knowledge of what the words and sentences mean. Frequent reading aloud by the student is helpful. This reading may well replace some of the lecturing by the teacher. Well-placed questions should insure a critical attitude and an attempt to retain and relate the thought. An easy, informal atmosphere should prevail, as one of tenseness may defeat the purpose of this method.

Plane geometry has fewer and less complex applications than algebra in the usual undergraduate college courses. It can be more readily reviewed as needed. I was surprised, however, in taking up the trapezoidal rule with one class, that nobody knew the formula for the area of a trapezoid and most had forgotten the definition of this figure.

The chief value in the study of demonstrative plane geometry lies in resultant training in deductive reasoning and consequent increase in mathematical maturity.

Although meaning comes first, students should be encouraged both to reason and to remember important facts. The student who does not remember the principal identities of plane trigonometry, for example, will have plenty of trouble with integration.

One of my students raised an objection to the fact that I consistently lettered my triangles A, B, C. A former teacher had sold him so completely on the necessity for reasoning that he felt one should never use the same lettering twice. Mixing up the letters

is a good thing while learning, but uniform lettering is best after learning.

I like the following thought expressed by A. N. Whitehead on this matter: 'It is a perfectly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of operations which we can perform without thinking about them.'[3]

One of the difficulties involved in providing a variety of mathematical experiences is lack of time. Time may be saved in some cases by the substitution of planned incidental review for the formal kind. Sometimes more time than is needed may be used on a topic.

The time allotted to mathematics may be too short. Work in analytic geometry, calculus, statistics, and other subjects and topics that will be taken in college should not take up time needed to establish a good background for those who may go on in mathematics. They are more properly material for the second track students who will never get a chance to study them again.

It may be that more could be accomplished in fewer and longer periods. We have found this to be true in college freshman mathematics in summer school courses. The longer periods avoided the necessity for starting over again on topics unfinished from the preceding period.

If the textbooks in use do not provide sufficient variety of experience, they should be supplemented. Material may be obtained from notebooks kept in college courses, from the illustrative problems in college textbooks, and from other texts in the subject being studied. Older textbooks should not be disregarded as a possible source of needed exercises.

In conclusion, let us say that a good background for later work in and application of the mathematics being studied at any particular time requires experiences with a variety of situations, that the teacher should have had courses beyond those he is teaching in order to know what situations are likely to be encountered, and that time and materials should be made available for this purpose.

[3] A. N. Whitehead, *Introduction to Mathematics*, (New York: Henry Holt and Company, 1939) p. 61.

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THE ELEMENTARY THEORY OF NUMBERS

E. T. Bell

This subject is so extensive, and so intricate, that only a bare indication of a few of its simpler ideas can be given here.

1. *Divisibility.* The first concern is with the natural numbers, or the positive integers, or simply the numbers, 1, 2, 3, 4, $\cdots$, and their elementary properties with respect to divisibility. The number d divides the number n, written $d|n$, if there is a number q such that $n=qd$. If $d|n$, n is called a multiple of d. If $d|m$ and $d|n$, then $d|(rm+sn)$, where r, s are any numbers (as defined above), and this d is called a common divisor of m, n. If $a|n$ and $b|n$, n is called a common multiple of a, b. If $g|n$ and $d|g$, then $d|n$. From these definitions we have those of the greatest common divisor (G.C.D.) and least common multiple (L.C.M.) of m, n as in school arithmetic, but with a different twist. It is not the magnitude aspects that are emphasized but the divisibility relations. This is done because it is the procedure that generalizes to types of integers other than the numbers 1, 2, 3, 4, $\cdots$, specifically to algebraic integers. If g is a common divisor of m, n, and if every common divisor of m, n is a divisor of g, then g is the G.C.D. of m, n. If h is a common multiple of m, n, and if every common multiple of m, n is a multiple of h, then h is the L.C.M. of m, n. If the G.C.D. of m, n is 1, m, n are called coprime or relatively prime. A practical method for finding the G.C.D. of m, n proceeds from the theorem that integers q, r, $q\geq 0$, $0\leq r<n$ can be found such that $m=qn+r$. The product of the G.C.D. and L.C.M. of m, n is mn, so that the L.C.M. is known when the G.C.D. is. The properties of these two functions of m, n have analogues in the rudiments of the theory of lattices and Boolean algebra.

If p is a number other than 1 whose only divisors are 1 and p, p is called prime, or a prime. It is to be noted that number here is still natural number. It can be shown that every number other than 1 has at least one prime divisor, and that the total number of prime divisors of any number is finite. The next, which will not be proved, is less obvious than it seems. If the prime p divides the product mn of the numbers m, n then p divides at least one of m, n. This is used in proving that there is no largest prime. For assume that P is the largest prime. Then the product $2\cdot 3\cdot 5\cdots P$ of all the primes being divisible by each of them, the number $2\cdot 3\cdot 5\cdots P+1$ is divisible by none of them. Hence it is

either a prime, or is divisible by a prime greater than P. Either possibility contradicts the assumption. This theorem is also stated as the number of primes is infinite.

The fundamental theorem of arithmetic asserts that, apart from permutations of the factors, a number other than 1 is uniquely a product of primes. Proof is by what precedes and a contradiction. It follows that a number n other than 1 is representable uniquely (up to permutations) in the form $p_1^{a_1} \cdots p_s^{a_s}$, where $p_1, \cdots, p_s$ are different primes and $a_1, \cdots, a_s$ are numbers (natural numbers). The divisors of this n are all the numbers of the form $p_1^{b_1} \cdots p_s^{b_s}$, where $0 \le b_i \le a_i$, $i = 1, \cdots, s$; whence it follows immediately that n has $(a_i + 1) \cdots (a_s + 1)$ divisors and that their sum is

$$\frac{p_1^{a_1+1} - 1}{p_1^{-1}} \cdots \frac{p_s^{a_s+1} - 1}{p_s^{-1}} .$$

Another function, $\phi(n)$, of the divisors of n, called Euler's function or the totient of n, is important in the study of divisibility. It is defined as the number of numbers that do not exceed n and are prime to n, and it is fairly easy to prove that, for $n > 1$,

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_s}\right).$$

Either by convention or from the definition, $\phi(1) = 1$. It is a simple exercise to prove that the sum of the totients of all the divisors of n is equal to n. For example, $n = 6$; the divisors of 6 are 1, 2, 3, 6, and $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(6) = 2$; $1 + 1 + 2 + 2 = 6$.

By a few obvious changes in the wording, the results of this section can be carried over to the set of all integers, $\cdots -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots$. The introduction of the negatives and zero is a convenience. It can be, and has been, avoided, but at the cost of intolerable prolixity. The natural numbers are the positive integers. It will be clear in a given context whether an integer, unqualified, is restricted to be positive.

2. *Congruence*. This concept leads to a vast theory having some striking analogies with the theory of algebraic equations. The integers a, b are said to be congruent with respect to the integer modulus m, or congruent modulo m, and this is written

$$a \equiv b \mod m,$$

if $m \mid (a - b)$ or, what is the same, if a, b leave the same remainder on division by m. Since a non-negative integer must leave precisely one

of 0, 1, $\cdots$, $m-1$ as remainder on division by m, the set of all integers (positive, zero, negative) falls into precisely m mutually exclusive sets, called the residue classes modulo m, with respect to congruence mod m, all the integers in a particular class being congruent to one another. These residue classes may be denoted by $C_0, C_1, \cdots, C_{m-1}$, all the integers in C_j being congruent to j mod m. They have an interesting and simple algebra, which may be left to the curiosity of the reader.

The relation of congruence is symmetric, reflexive, and transitive: if $a \equiv b$ mod m, then $b \equiv a$ mod m; $a \equiv a$; if $a \equiv b$ mod m and $b \equiv c$ mod m, then $a \equiv c$ mod m. These are immediate from the definition, and the next follow readily. If $a \equiv b$ mod m and $f \equiv g$ mod m, then $a \pm f \equiv b \pm g$ mod m, and $af \equiv bg$ mod m. By repeated applications of these, if $P(x)$ is a polynomial in x with integer coefficients, $a \equiv b$ mod m implies $P(a) \equiv P(b)$ mod m.

So far the analogy between equations and congruences is close. In the next there is a radical difference. The equation $P(x) = 0$ is completely solvable in the field of complex numbers; there may be no integer x such that $P(x) \equiv 0$ mod m, that is, the congruence may have no roots. The analogy is partially restored however in the theorem that if m is prime, the congruence cannot have more incongruent (distinct modulo m) roots than its degree. Again, a common factor may be cancelled from all the coefficients of an equation, while the example $3x \equiv 3y$ mod 6, or $3(x-y) \equiv 0$ mod 6, shows that if 3 is cancelled, then $x-y \equiv 0$ mod 2. Generally, if $ax \equiv ay$ mod m, and if d is the G.C.D. of a, m, then $x \equiv y$ mod m/d. This is used in the following proof of what has been called a cornerstone of the theory of numbers.

Let b be an integer prime to the positive integer m; write the totient $\phi(m) = s$, and denote by $b_1, \cdots, b_s$ the s positive integers not exceeding m and prime to m. By a contradiction it is proved that the s products $bb_1, \cdots, bb_s$ are congruent modulo m, in some order, to $b_1, \cdots, b_s$. Hence $bb_1 \cdots bb_s \equiv b_1 \cdots b_s$ mod m. The product $b_1 \cdots b_s$, being prime to m, may be cancelled. Thus $b^{\phi(m)} \equiv 1$ mod m. If m is the prime p, $\phi(p) = p-1$. Hence if b is not divisible by the prime p, $b^{p-1} - 1$ is divisible by p. These two theorems and their proofs are typical of many in the theory of numbers. Each might be inferred from empirical evidence; each is simply intelligible; the device which yields the proofs might—as it did—elude the ingenuity of first-rate mathematicians for many years. The result $p \mid (b^{p-1} - 1)$, p prime, b not divisible by p, is called Fermat's theorem. It is of importance in algebra, for example in the theory of binomial equations.

Fermat's theorem leads to the subject of primitive roots. It has been seen that $b^{\phi(m)} \equiv 1$ mod m for b prime to m. There may be a positive exponent $e < \phi(m)$ for which $b^e \equiv 1$ mod m. If e is the least exponent for which the congruence holds, b is said to appertain to the exponent e modulo m. It is quite easy to show that there are exactly $\phi(e)$

incongruent (distinct modulo p) numbers modulo the prime p appertaining to the exponent e, where e is any divisor of $p-1$. For $e = p-1$, there are thus $\phi(p-1)$ roots appertaining to the exponent $p-1$; these are called the primitive roots of p. By a theorem stated earlier concerning ϕ, $\Sigma\phi(e)$, where e ranges over all the divisors of $p-1$, is equal to $p-1$. This may be used in proving the theorem concerning primitive roots.

Another application of the result $b^{\phi(m)} \equiv 1 \bmod m$ is to the solution of the congruence $ax \equiv b \bmod m$, where, by an earlier theorem, a, m may be taken coprime. There is the evident but not very practical solution $x = ba^{\phi(m)-1}$. But the congruence may be replaced by the equivalent equation $ax + my = b$. Euclid's algorithm for the G.C.D. leads to the conversion of a rational fraction into a continued fraction. The penultimate convergent to the continued fraction for a/m (or m/a, whichever is the smaller in absolute value) furnishes a solution of $ax' + my' = 1$; $x = bx'$, $y = by'$ is then a solution of the equation.

A corollary to Fermat's theorem furnishes a necessary and sufficient but unusable condition that a given number be prime. For that $x^{p-1} - 1 \equiv 0 \bmod p$ (prime) has exactly $p-1$ incongruent roots mod p, namely $x = 1, 2, \cdots, p-1$, is the content of Fermat's theorem. Hence, identically in x,

$$x^{p-1} - 1 \equiv (x-1)(x-2)\cdots(x-p+1) \bmod p.$$

For $x = 0$ and the prime p odd, this becomes $(p-1)! + 1 \equiv 0 \bmod p$, which evidently holds also for $p = 2$. This is Wilson's theorem: for every prime p, p divides $(p-1)! + 1$. The converse is readily proved. Hence a necessary and sufficient condition that p be prime is that p divide $(p-1)! + 1$. Fermat's theorem also has a converse. If there is an integer x such that $x^{m-1} \equiv 1 \bmod m$, while for no exponent $e < m-1$ is $x^e \equiv 1 \bmod m$, then m is prime. This has proved usable in certain tests for primes.

With increase in the degree of congruences the difficulties in solving them, or even in deciding whether they have solutions, increases rapidly. Many of the questions raised by such problems are still far from answered. Congruences of the second degree in one variable (or indeterminate) x lead to what Gauss called the gem of arithmetic. To exhibit it, some definitions are necessary. If r, m are coprime integers such that the congruence $x^2 \equiv r \bmod m$ has a solution x, r is called a quadratic residue of m; if there is no such x, r is a quadratic non-residue. Legendre's symbol $(\frac{r}{m})$ denotes +1 or −1 according as r is a quadratic residue or a quadratic non-residue of m. Let p, q be odd primes. The following are almost immediate.

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \quad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}};$$

the second is obtainable from Wilson's theorem, among several other ways.

The next, the law of quadratic reciprocity, the gem to be displayed, is not easy to prove unless one has been shown how:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{1}{2}(p-1) \times \frac{1}{2}(q-1)}$$

There are many proofs.

Congruences of degree higher than the second suggest the investigation of further reciprocity laws. All this belongs to the advanced part of the theory of numbers where even a competent man might spend the better part of his life without getting very far.

From the few theorems described, it may be surmised that a very considerable part of the theory of numbers is concerned with primes and their properties. Some apparently sensible questions concerning primes have not been answered; others have, more or less. For example, how many primes are there less than x still awaits a usable solution, if one is attainable. Asymptotically, the number of primes not exceeding x is $x/\log x$. Only one unsolved problem on primes will be mentioned here, because it is the only one of many that conceivably might yield something to elementary ingenuity. Is the number of so-called Fermat primes $2^{2^n} + 1$ finite, or is it infinite? For the connection of these primes with cyclotomy, see any history of mathematics. It has been proved that if a, b are coprime, the arithmetic progression $ax + b$, $x = 1, 2, \cdots$ contains an infinity of primes. For many special a, b strictly elementary proofs for the corresponding progressions have been given, and it seems reasonable to suppose that sufficient elementary ingenuity would dispose of any particular pair. However, the proof of the general theorem is by advanced and somewhat delicate analysis. A general elementary proof is a desideratum—either that or a proof that no such proof is possible.

3. *Forms.* Fermat proved by his method of descent that a prime p of the form $4n + 1$ can be represented in essentially one way only as a sum of two integer squares. This was one root of the vast and still expanding arithmetical theory of forms. Form, without qualification, means a homogeneous polynomial with integer coefficients. A capital problem is to determine what integers are representable in a given form when integer values are assigned to the variables (indeterminates $x, y, z, \cdots$) in the form. If the form is $f(x, y, z, \cdots)$, the problem amounts to solving $f(x, y, z, \cdots) = m$, for m given, in integers $x, y, z, \cdots$, or if for certain m there are no integer solutions, proving that there are none. An example of extreme difficulty is to prove or disprove the conjecture that zero is not represented in the form $x^n + y^n + z^n$ by integers x, y, z all different from zero when $n > 2$. The discussion of questions of this order of difficulty is likely to demand the invention of new methods and the discovery of new principles as this one, Fermat's, did in giving rise to the theory of ideals in algebraic numbers.

Much has been done for the case of forms of the second degree, and for such forms in two variables there is a reasonably complete theory.

For three or more variables the difficulties increase rapidly. In this field the apparatus of linear homogeneous substitutions with integer coefficients and determinant +1 or −1 is one of keys to the main problems. Forms transformable into one another by such substitutions are said to be equivalent. A given form is transformed into a simpler equivalent; if the theory of representations in the second form can be obtained, that in the first follows. Much less has been done for forms of degree higher than the second.

A famous result for forms of the second degree in four variables states that every integer is a sum of four integer squares (zero included as a possibility). Another, for forms of the second degree in three variables, states that any integer not of the form $4^k(8n+7)$ is a sum of three integer squares with no common factor >1. The first of these was one germ of the current arithmetic of quaternions with integer coefficients. It may also have inspired Waring's famous conjecture (?) that every positive integer is a sum of a fixed number, $g(k)$, of kth powers of positive or zero integers. Thus $g(4)=4$; it is known that $g(3)=9$, and the general theorem—the existence of $g(k)$ for all k but not its exact value for all k, notably for $k=4$—was proved in the present century by the efforts of several mathematicians using different methods, none simple.

Another striking theorem, stated by Fermat in 1636 and proved by Cauchy in 1815, asserts that every positive integer is a sum of $m+2$ so-called polygonal numbers $\frac{1}{2}m(x^2-x)+x$. Cauchy proved that all but four of the $m+2$ can be taken 0 or 1. For $m=2$ this is the four-square theorem.

The last suggests a seemingly simpler type of problem. In how many ways can a positive integer be represented as a sum of unrestricted positive integers, or as a sum of restricted positive integers, such as all odd? Modern work on this problem employs advanced analysis in the derivation of asymptotic formulas. Such formulas are of frequent occurrence in statistical mechanics, where sometimes exact computation appears to be humanly impossible, not to say unnecessary. This problem was one of the main sources, in Euler's work, of the theory of the elliptic theta functions.

Another vast domain, Diophantine analysis, may be mentioned here, although it is not usually thought of as belonging to the theory of forms, possibly because a general theory of forms is yet to be created. But it appears naturally enough when the restriction of homogeneity is removed. The main problem here is to devise a method for deciding when a given equation with integer (or rational) coefficients has integer (or rational) solutions, and if it has, to find them all. Little of any generality has been done in this direction.

4. *Algebraic numbers.* As this sketch began with divisibility, it may fittingly close with a mere mention of a generalization of the theory as sketched. A root of an irreducible algebraic equation of degree n is called an algebraic number of degree n. If all the coefficients are (rational) integers, and the leading coefficient is 1, the roots are called algebraic integers. For some classes of such integers the fundamental theorem of arithmetic fails, and an integer may be a product of prime

integers in more than one way. The fundamental theorem was restored by the introduction of ideals. For anyone wishing to follow this development excellent texts, mostly in German, are available. It is interesting and instructive in studying this subject to observe how some of the seminal concepts of modern abstract algebra appeared first in the theory of algebraic numbers. This is characteristic of the entire subject. Although its direct contributions to the sciences have been few compared with those of other departments of mathematics, the theory of numbers has supplied those departments with methods, problems, and ideas that might not otherwise have been imagined.

The April, 1949 issue of the *Annals of Mathematics* contains elementary proofs by Atle Selberg of the prime number theorem and Dirichlet's theorem on the primes in an arithmetic progression. 'Elementary' is a relative term. For the prime number theorem the sense is that practically no analysis except the simplest properties of the logarithm is used. For Dirichlet's theorem complex characters mod k are obviated and only finite sums are considered.

California Institute of Technology E. T. Bell

MATHEMATICAL MISCELLANY

Edited by

Marion E. Stark

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other 'matters mathematical' will be welcome. Address: MARION E. STARK. Wellesley College, Wellesley, 81, Mass.

The letter of the month:

I should like to propose a query for the *Mathematical Miscellany* you edit in the *Mathematics Magazine.* How many texts on Trigonometry have been published in the United Staees since 1899? Editions later than a first are to be counted as new tests. I believe the answer can be found by any industrious librarian on consulting the standard professional library equipment.

If and when this query is answered, another may be proposed. Why so many?

Yours sincerely,

E. T. Bell

California Institute of Technology.

Editorial Comment:

Answers to, and comments on, the above are eagerly desired. And shall we limit it to Trigonometry? All in favor of widening the scope of this discussion will answer Aye. The Ayes have it.

Next we have a song written for a mathematics dinner by R. Lariviere of Chicago. The tune is 'Let me call you Sweetheart.' Now, everybody join in.

Oh, Miss Intuition, I'm in love with you.
You are always steady and I hold you true.
Let me call you Logic — one and one make two —
Oh, Miss Intuition, I'm in love with you.

I've liked wave mechanics, n-dimensions too.
Hyperbolic geodesics, they have thrilled me through.
But, Miss Intuition, they are not like you.
Let me call you Logic. I'm in love with you.

Approximations to Square Roots

Two methods of taking approximate square roots are widely used.

1. Let a be an approximate value of $\sqrt{x}$. Then a better value of $\sqrt{x}$ is $\left(\frac{x}{a} + a\right) \div 2$. This is equivalent to using $y = \frac{(x + a^2)}{2a}$ in place of the curve $y = \sqrt{x}$, and as the former may be written in the form $\frac{(y - a)}{(x - a^2)} = \frac{1}{2a}$, this method is equivalent to using the tangent to $y = \sqrt{x}$ at the point (a^2, a). Since the parabola, $y = \sqrt{x}$, always lies below its tangent, this method always gives too large a result. This fact may be shown also by transforming $\frac{(x + a^2)}{2a} - \sqrt{x}$ to the fraction $\frac{(x + a^2 - 2a\sqrt{x})}{2a}$, whose numerator is the square of $(a - \sqrt{x})$ and is therefore always positive.

It is worth while to note that this is the approximation to which we are led by the binomial theorem. For if we have $x = a^2 + z$, we get $\sqrt{x} = a + \frac{z}{2a} - \frac{z^2}{8a^3} \ldots$, whose first two terms give $\sqrt{x} = a + \frac{(x - a^2)}{2a}$.

2. Take a and $a + 1$, the two integers nearest to $\sqrt{x}$, and interpolate. This is usually illustrated thus:

$$\sqrt{4} = 2, \quad \sqrt{5} = 2\tfrac{1}{5}, \quad \sqrt{6} = 2\tfrac{2}{5}, \quad \sqrt{7} = 2\tfrac{3}{5}, \quad \sqrt{8} = 2\tfrac{4}{5}, \quad \sqrt{9} = 3.$$

The interpolation is equivalent to taking a chord instead of the curve $y = \sqrt{x}$, and since the chord is always below the parabola, this method always gives too small a root.

The algebraic formula for this method is obtained by taking two numbers, a and $a + 1$, between whose squares x lies, and forming the equation of the line through (a^2, a) and $([a + 1]^2, a + 1)$, which is $\frac{(y - a)}{(x - a^2)} = \frac{1}{(2a + 1)}$. To compare this with the other method, reduce to the form $y = a + \frac{(x - a^2)}{(2a + 1)}$. We may show that this is too small by transforming $a + \frac{(x - a\ )}{(2a + 1)} - \sqrt{x}$ to the fraction

$$\frac{(2a^2 + a + x - a^2 - 2a\sqrt{x} - \sqrt{x})}{(2a + 1)},$$

whose numerator may be written $(a - \sqrt{x})^2 + (a - \sqrt{x})$, or $(a + 1 - \sqrt{x})(a - \sqrt{x})$: the first factor is positive and the second is negative.

The two forms, $a + \frac{(x - a^2)}{2a}$ and $a + \frac{(x - a^2)}{(2a + 1)}$ are particularly

valuable because of their resemblance and because they shut in the value of $\sqrt{x}$ between limits. The fact that a curve lies between its chord and its tangent is often useful in making approximations.

Tufts College William R. Ransom

We think the following article will be of interest to readers. It is said to be the work of a fifteen year old boy. Would that more of this sort of thing might come out of our high schools. The boy is a second year student at the Stuyvesant High School in New York City.

In the article in formula VI the superscript on the second sigma should be x rather than n. Mr. Towber's $c(x)$ is the Kronecker delta, $\delta(c,x)$. He has, therefore, one representation of $\delta(c,x)$.

A Formula for the nth Prime

The simplest way to obtain a function of n that will yield the nth prime (where n is any whole number) is to compound suitably simpler functions of n which have certain special properties. In order to determine the form these are to have, a definite plan of attack must first be formulated. The one I chose is here outlined.

One way of regarding p_n, the nth prime, is as the sum of an infinite series all of whose terms are zero except the one term p_n. Thus, if we can construct a function $p_n(x)$ differing from zero only when $x = \widehat{p_n}$ and equalling unity for that value of x, we can express p_n in the form $\sum_{i=1}^{\infty} i\widehat{p_n}(i)$.

Only three auxiliary functions will be needed in the construction of $\widehat{p_n}(x)$

(a) A function $\hat{c}(x)$ such that
(1) if $x = c$, $\hat{c}(x) = 1$, and
(2) if $x \neq c$, $\hat{c}(x) = 0$.

(b) A function $D(x)$, giving the number of divisors of x.

(c) A function $L(x)$, giving the location of x in the prime sequence, if x is prime. For non-prime values of the argument, our function is to vanish.

I shall discuss these functions in order:
first, the function

I. $c(x) = \lim \dfrac{z}{z - c + x}$ has the properties of $c(x)$ in (a) above, for

(1) if $x = c$, $\lim\limits_{z\to 0} \dfrac{z}{z - c \quad x} = \lim\limits_{z\to 0} \dfrac{z}{z} = 1$, and

(2) if $x \neq c$, (say $x - c = d$ which is $\neq 0$), then

$$\lim_{z\to 0} \frac{z}{z + (x - c)} = \lim_{z\to 0} \frac{z}{z + d} = \frac{0}{0 + d} = 0.$$

Secondly, since $\sin \pi x = 0$ if and only if x is a whole number, the function $\hat{c}(\sin \pi x)$[where $c = 0$, and $\hat{c}(x)$ is defined as in (a)] is evidently 1 for integral values of x, and zero for all other values. This function may be put in the form $\lim_{z\to 0} \frac{z}{z + \sin \pi x}$ by direct substitution in the results of I above. Since i divides m if and only if $\frac{m}{i}$ is a whole number, we have the useful result:

II. $\lim_{z\to 0} \frac{z}{z + \sin \frac{\pi m}{i}}$ one or 0, according as i does or does not divide m. Since $\lim_{z\to 0} \frac{z}{z + \sin \frac{\pi m}{i}}$ vanishes for all i except divisors of m, any terms of this form in a sum will drop out for non-divisors i of m. Thus, since there are exactly as many ono-zero terms in the sum $\sum_{i=1}^{m} \lim_{z\to 0} \frac{z}{z + \sin \frac{\pi m}{i}}$ as there are divisors of m, and each of these non-zero terms equals unity, it follows that this sum yields exactly the number of divisors of m, that is, if $D(x)$ is defined as in (b) above, we have

$$\text{III.} \quad D(m) = \sum_{i=1}^{m} \lim_{z\to 0} \frac{z}{z + \sin \frac{\pi m}{i}}$$

A number is prime if it has exactly two distinct divisors (itself, unity, and no others). Therefore, by combining equations I and III, we can show easily that p is prime or non-prime according as

$$\text{IV.} \quad \lim_{z\to 0} \frac{z}{z - 2 + \sum_{i=1}^{p} \lim_{r\to 0} \frac{r}{r + \sin \frac{\pi m}{i}}} = 1 \text{ or } 0.$$

Reasoning exactly as we did in the preceding analysis, we may now show that

V. $\sum_{p=1}^{m} \lim_{z\to 0} \frac{z}{z - 2 + \sum_{i=1} \lim_{r\to 0} \frac{r}{r + \sin \frac{\pi p}{i}}}$ equals the number of primes not exceeding m. Now, if m is prime, this function is exactly $L(m)$, the function defined in (c). On the other hand, if m is non-prime, our function does not vanish. In order to ensure its evanescence for non-primes, we shall multiply by the function IV. For non-primes,

this assumes the form (a finite number) times zero, and the product vanishes. For primes, however, it takes on the form (1) $[L(x)] \equiv L(x)$. In other words, we have obtained the following expression for $L(x)$:

$$\text{VI. } L(x) = \left[\lim_{z\to 0} \frac{z}{z-2+\sum_{i=1}^{x} \lim_{r\to 0} \frac{rG}{r+\sin\frac{\pi x}{i}}}\right]\left[\sum_{p=1}^{n} \lim_{z\to 0} \frac{z}{z-2+\sum_{i=1}^{p} \lim_{r\to 0} \frac{r}{r-2+\sin\frac{\pi p}{i}}}\right]$$

Having obtained our auxiliary function, we can now write $p_n(x)$. Indeed, referring to the definition of $p_n(x)$, we see that $p_n(x) = \lim \frac{z}{z - n + L(x)}$ where $L(x)$ is given by VI.

Finally, we obtain the nth prime in the form

$$p_n = \sum_{i=1}^{\infty} i p_n(i) =$$

$$\sum_{i=1}^{\infty} i \lim_{z\to 0} \frac{z}{z-n+\left[\lim_{r\to 0} \frac{r}{(r-2)+\sum_{j=1}^{i} \lim_{s\to 0} \frac{s}{s+\sin\frac{\pi i}{j}}}\right]\left[\sum_{k=1}^{i} \lim_{r\to 0} \frac{r}{(r-2)+\sum_{j=1}^{k} \lim_{s\to 0} \frac{s}{s+\sin\frac{\pi k}{j}}}\right]}$$

Jacob Towber

Dear Professor James;

I have been thinking some more about the project I tentatively proposed in my last letter since I learned that you had already been cogitating along similar lines,— something of a Mathematical Forum or Mathematical Soundingboard. This department would publish letters either signed or anonymous dealing with nontechnical matters in the mathematical world. The letters could be aimed at improving certain mathematical situations but should be strictly impersonal. We could get a freer expression if we allowed anonymous letters but of course there are objections to this. My guess is that subscribers alone would provide more letters than we could publish. If the letters are non-tecnical then even people with no mathematical training could find them interesting and enjoyable. My talks with colleagues convince me that most mathematicians have something that they would like to express an opinion about and it is entirely possible that some good might come of such a program. Once a few controversial matters appear in print, I believe we would get a lot of action in the form of expressions of opinion. What would you think of asking

our readers if they would like to have such a department?

As ever

H. V. Craig

I would like very much to have the reactions of our readers to this suggestion.

Glenn James

(Continued from inside back cover)

Harold Everett Bowie, Associate Professor of Mathematics and Dept. Head, American International College, was born on January 23, 1901 in Durham, Maine, and attended the University of Maine (B.A.'28; M.A.'32). He was administrator and teacher in the public secondary schools of Maine from 1921 to 1936 and was instructor in Mathematics at the University of Maine before taking a position at American International College in 1938. His chief mathematical interest is in analysis.

Biographical sketches of *E. T. Bell* and *Pedro A. Piza* were published in vol. XXI, no. 2. *Gordon Raisbeck's* sketch will appear in the next issue.

OUR CONTRIBUTORS

Harry Bateman was born in Manchester, England in 1882. From 1900-05 he attended Trinity College, Cambridge (B.A.'03, M.A.'06) and won a fellowship there in 1905. After teaching at Liverpool and Manchester, Prof. Bateman came to America and spent several years at Johns Hopkins as Johnston Scholar and lecturer. In 1917 he was appointed Professor of Mathematics, Physics and Aeronautics at the California Institute, where he remained until his death in 1946. A distinguished mathematician and mathematical physicist, Dr. Bateman received many honors, including election to the National Academy, Fellow of the Royal Society, etc. for his work in electrodynamics, hydrodynamics, special functions and other fields. (For a more adequate sketch see the article by Murnaghan, Bull. A.M.S. 54 (1948) pp. 88-103).

Joseph Barnett, Associate Professor of Mathematics, (Emeritus), Oklahoma A. and M. College, was born at Jarvisville, West Virginia in 1883. After teaching in the rural schools of W. Virginia, he attended Virginia Wesleyan College and West Virginia University (A.B.'14) and later on, Columbia (M.A.'23). Mr. Barnett joined the faculty of Oklahoma A. & M. in 1925 and was promoted to associate professor in [illegible]. Geometry is his special interest.

Clarence De Witt Smith, Assoc. Prof. of Statistics, University of Alabama, was born at Jackson, Miss. in 1891. Prof. Smith went to Mississippi College (A.B.'15) and was professor of mathematics at Louisiana College from 1919-30. After completing his graduate studies at Iowa State (M.S.'25, Ph.D.'28) he became prof. of math. and statistics at Mississippi State (1930-46) and was appointed to his present position two years ago. A member of the Inst. of Math. Statistics and the Math. Assn., Prof. Smith has published in the fields of geometry, probability and statistics.

Vincent O. [illegible], Ass't Prof. of Math., College of the Holy Cross, was born in [illegible], Massachusetts, in 1916. He attended Providence College (B.S.'[illegible]) and the Catholic Univ. of America, (Ph.D.'42). He spent a year at the David Taylor Model Basin and the O. S. R. D. working on research projects for the Navy Dept. After teaching for a year at Hamilton College he came to Holy Cross in 1945. His special mathematical interest is Inversive Geometry. He is a member of the Amer. Math. Soc., the Math. Ass'n. of Amer. and the A.A.A.S.

Sister M. Philip [illegible], O.P., Associate Professor of Mathematics, Rosary College, [illegible] Forest, Illinois, was born in Sioux City, Iowa in 1905. She is a graduate of Rosary College (A.B.,'27), the University of Wisconsin ([illegible]), and the Catholic University of America (Ph.D.'[illegible]). She is a member of the American Math. Soc., and of Kappa Gamma Pi, National Honor Society of Catholic Women's Colleges. Sister Philip is especially interested in Algebra and Group Theory.

(Continued on page [illegible])